

Expressiveness and Computational Complexity of Geometric Quantum Logic

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Abstract. Quantum logic generalizes, and in dimension one coincides with, Boolean logic. We show that the satisfiability problem of quantum logic formulas is \mathcal{NP} -complete in dimension two as well. For higher higher-dimensional spaces \mathbb{R}^d and \mathbb{C}^d with $d \geq 3$ fixed, we establish quantum satisfiability to be polynomial time equivalent to the real feasibility of a multivariate quartic polynomial equation: a problem well-known complete for the counterpart of \mathcal{NP} in the Blum-Shub-Smale model of computation lying (probably strictly) between classical \mathcal{NP} and PSPACE. We finally investigate the problem over *indefinite* finite dimensions and relate it to the real feasibility of quartic *noncommutative* *-polynomial equations.

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1 Introduction

In view of recent inflationary occurrences of the word “quantum” [Flem60], recall that the origins of its rise reside in particle physics. Putting a mathematical foundation to the new theories describing the paradoxical (yet very real) effects of quantum physics [Mack63] has led to Quantum Logic as a generalization of classical Boolean logic in that the distributive law may fail in some cases [BiVN36]:

Definition 1. Fix some integer $d \in \mathbb{N}$ and let $\mathbb{F} \subseteq \mathbb{C}$ denote a field (popular cases being e.g. $\mathbb{F} = \mathbb{C}$ itself, $\mathbb{F} = \mathbb{A}$ algebraic numbers, $\mathbb{F} = \mathbb{R}$ real numbers, $\mathbb{F} = \mathbb{Q}$ rationals, and $\mathbb{F} = \mathbb{A} \cap \mathbb{R}$ algebraic reals). The *quantum logic* of \mathbb{F}^d consists of the set $\text{Gr}(\mathbb{F}^d) := \bigcup_{i=0}^d \text{Gr}_i(\mathbb{F}^d)$ of all subspaces of \mathbb{F}^d of dimension** $0 \leq i \leq d$ (i.e. the Grassmannian stripped of its topology, and instead) equipped with the following connectives:

- $\neg : \text{Gr}_i(\mathbb{F}^d) \rightarrow \text{Gr}_{d-i}(\mathbb{F}^d)$, $P \mapsto P^\perp := \{\vec{y} \in \mathbb{F}^d : \langle \vec{y}, \vec{x} \rangle = 0 \ \forall \vec{x} \in P\}$
- $\wedge : \text{Gr}(\mathbb{F}^d) \times \text{Gr}(\mathbb{F}^d) \rightarrow \text{Gr}(\mathbb{F}^d)$, $(P, Q) \mapsto P \cap Q$
- $\vee : \text{Gr}(\mathbb{F}^d) \times \text{Gr}(\mathbb{F}^d) \rightarrow \text{Gr}(\mathbb{F}^d)$, $(P, Q) \mapsto P + Q$.

We abbreviate $0 := \{0\} \in \text{Gr}(\mathbb{F}^d)$ and $1 := \mathbb{F}^d \in \text{Gr}(\mathbb{F}^d)$.

Note that $P + Q = \{p + q : p \in P, q \in Q\} = \text{lspan}(P \cup Q)$.

Fact 2. $(\text{Gr}(\mathcal{H}), \neg, \vee, \wedge, \{0\}, \mathcal{H})$ constitutes a *modular ortholattice*.

Here an ortholattice $(\mathcal{L}, \neg, \vee, \wedge, 0, 1)$ is a commutative, associative algebra w.r.t. “ \vee ” and “ \wedge ” (i.e. a lattice) with $A \wedge (A \vee B) = A = A \vee (A \wedge B)$ (hence $A \vee A = A = A \wedge A$) and $A \vee \neg A = 1$ and $A \wedge \neg A = 0$ and $\neg \neg A = A$ and such that the *de Morgan laws* hold: $\neg(A \vee B) = \neg A \wedge \neg B$ and $\neg(A \wedge B) = \neg A \vee \neg B$. The de Morgan laws are well-known [Bera85, SECTION II.1] equivalent to the partial order “ \preceq ” on \mathcal{L} , defined by $A \preceq B :\Leftrightarrow A = A \wedge B$, having $A \preceq B \Rightarrow \neg B \preceq \neg A$. An ortholattice is *modular* if it holds

$$A \preceq C \quad \Rightarrow \quad A \vee (B \wedge C) = (A \vee B) \wedge C . \quad (1)$$

Definition 1 and Fact 2 extend to the set $\text{Gr}(\mathbb{F}^\infty)$ of co-/finite dimensional subspaces of the (or any other) separable Hilbert space $\mathbb{F}^\infty := \ell^2(\mathbb{F}) = \{(x_n)_{n \in \mathbb{N}} : x_n \in \mathbb{F}, \sum_n |x_n|^2 < \infty\}$ of square-summable sequences over either $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$ (for reasons of metric completeness).

Remark 3. Definition 1 can furthermore be extended to the set of closed subspaces of some separable Hilbert space \mathcal{H} , where $P \vee Q = \overline{P + Q}$ now has to be understood as the topological closure of $P + Q$:

Consider closed subspaces $P := \{(x_1, 0, x_3, 0, \dots, x_{2n+1}, 0, \dots) : x_n \in \mathbb{F}, \sum_n |x_n|^2 < \infty\}$ and $Q := \{(x_1, x_1/1, x_3, x_3/3, \dots, x_{2n+1}, x_{2n+1}/(2n+1), \dots) : \dots\}$ of ℓ^2 . Then $(0, 1, 0, 1/3, \dots, 0, 1/(2n+1), \dots) \in \ell^2$ belongs to $\overline{P + Q}$ but not to $P + Q$.

This will still constitute an ortholattice but fails the modular law (1); however it still satisfies the weaker *orthomodular law* $A \preceq C \Rightarrow A \vee (\neg A \wedge C) = C$.

The present work’s focus is on the finite-dimensional cases; cf. Section 1.4.

** We adopt the affine notion of dimension because it coincides with the height of the induced lattice as opposed to the projective dimension

1.1 Motivations

We find quantum logic a promising object of study for various reasons and to various communities [CMW00,EGL07]. Our presentation tries to be accessible to all of them; in particular some explanations below which may seem shallow to computational complexity theorists are in fact aimed for theoretical physicists.

1.1.1 Physically, it arises from Quantum Mechanical reality. More formally, a classical 0/1-valued observable p (*‘property’*) corresponds in quantum mechanics to a projection operator \mathbb{P} on some Hilbert space \mathcal{H} of states; which in turn corresponds uniquely to a closed subspace P of \mathcal{H} , namely $P = \text{range}(\mathbb{P})$ and \mathbb{P} being the orthogonal projection onto P [Neum55,Jack06]. That is, P amounts to the collection of states where property \mathbb{P} is present; $\neg P$ where it is absent; $P \wedge Q$ where both \mathbb{P} and \mathbb{Q} are present; and $P \vee Q$ where at least one of \mathbb{P} and \mathbb{Q} are. The subtleties of Quantum Mechanics inherently lead to so-called non-pure states (e.g. a photon linearly polarized at 45°) where neither a property (e.g. linear polarization at 0°) nor its complement (linear polarization at 90°) need be absent. As a consequence, the classical law $Q = (Q \wedge P) \vee (Q \wedge \neg P)$ may fail in Quantum Logic. And the uncertainty exhibited by measurements of complementary observables (e.g. linear polarization at 45° and at 0° ; or position and momentum) is inherent to Quantum Mechanics according to the famous Kochen-Specker [KoSp67] and Bell’s Theorems.

1.1.2 Algebraically, EMMY NOETHER had initiated and propagated the trend of *algebraization* in mathematics: Instead of opaque proofs by repeated ϵ – δ type arguments (so-called *hard analysis*), capture the essential properties once into elementary axioms and then obtain results and insight by purely logical deduction. This (meta-)approach has become known as *synthetic*. For functional analysis on Hilbert space, it amounts precisely to Quantum Logic; cf. Fact 2 above and Section 1.3 below.

Note, however, that we deliberately consider geometrical (and in particular dimensional) as opposed to axiomatic quantum logic [Wilc09].

1.1.3 Geometrically, Quantum Logic captures the combinatorial aspects of the Grassmannian; compare the proof of the aforementioned Kochen-Specker Theorem. It therefore belongs to the field of Convex and Combinatorial Geometry [GWS93,MnZi93,GoOR04]. And it exhibits some surprisingly rich structures in the seemingly simple topic of basic high-school linear algebra. In particular, all proofs are (essentially) elementary. For instance the failure of classical laws as mentioned above are instructive to reconstruct in the planar $\text{Gr}(\mathbb{F}^2)$; compare Example 7c) below. Quantum logic in *indefinite* dimensions (Section 5) corresponds to a kind of dimension-free or dimension-oblivious combinatorial geometry.

1.1.4 Intellectually, human thought is often reduced to Boolean 0/1 or black/white-type of arguments. For instance, educational brain teasers and riddles can often be solved by tedious exhaustion. We find it inspiring and exciting to practice non-Boolean deduction. For instance, *tertium non datur* obviously fails in (Constructive Mathematics and) Quantum Logic; see Definition 5d) below.

1.1.5 Logically, $\text{Gr}(\mathbb{F}^d)$ constitutes a structure which contains (and for $d = 1$ collapses to) the Boolean values $\text{false} = \{0\} = 0$ and $\text{true} = \mathbb{F}^d = 1$. Yet, starting with $d = 2$, $\text{Gr}(\mathbb{F}^d)$ also contains ‘intermediate’ values like $\text{lspan}(\frac{1}{r})$, $r \in \mathbb{Q}$. This many-valuedness resembles probabilistic/fuzzy logic, and more generally Gödel Logics, which ‘interpolate’ between 0 (impossibility) and 1 (certainty) with values $r \in [0, 1]$ (likelihood). However the latter truth values are pairwise comparable; whereas, due to the complex-valued background of Quantum Mechanics where only the magnitude of the wave function amounts to the probability and the phase gives rise to astonishing interference effects, intermediate values in Quantum Logic generally are not comparable but induce only a partially ordered set lacking distributivity.

1.1.6 The Computational Complexity of problems involving Boolean formulas and circuits has been closely investigated in Computer Science and captured in standard complexity classes of (Turing-) in-/tractability; cmp. [Papa94]. For instance, evaluation at a given assignment of Boolean inputs is \mathcal{P} -complete; satisfiability (existence of an assignment yielding value **true**) is \mathcal{NP} -complete; equivalence (identical values for all possible assignments) is coNP -complete; each further quantifier alternation climbs up one step in the polynomial hierarchy; and unboundedly many of them finally reach PSPACE -completeness. Since FEYNMAN it is an open question of whether quantum systems are able to solve in polynomial time problems that ordinary Turing machines require superpolynomially (or even exponentially) many steps for. Quantum computers for instance operate on several entangled states simultaneously and apply one final measurement. Quantum logic on the other hand describes operations on observables, that is sequences of measurements, and have been proposed as an alternative approach to exploit quantum mechanics for computational purposes [Pyka00,Ying05,PM07a].

Sections 3, 4.2, 4.3, and 4.5 below determine the computational complexity of satisfiability and equivalence problems in quantum logics $\text{Gr}(\mathbb{F}^d)$ as a generalization of the Boolean case $d = 1$.

1.1.7 Determining the Expressiveness of quantum logic formulas may be regarded as a common task in model theory, namely examining which structures can be embedded into quantum logic. Specifically we compare in Section 2.2 the numbers of distinct Boolean and quantum formulas, thus continuing a classical line of investigations of tackling differences and similarities between Boolean and quantum logic. Section 4.7 shows that, although ‘pure’ quantum logic in general cannot express the Boolean connectives (Remark 6), both quantum logic with constants and first-order quantified quantum can.

1.2 Expressiveness and Computational Complexity

are in fact closely related and serve as mutual tools in proofs: The ability to encode ring operations into 3D quantum logic (Corollary 49) for instance yields a reduction from real polynomial feasibility to quantum satisfiability, that is a lower complexity bound; whereas the inability to express Boolean connectives is crucial to the polynomial time algorithm deciding satisfiability of quantum formulas in conjunctive normal form (Section 3.1). Conversely, an analysis of polynomial-time Gaussian Elimination in the Blum-Shub-Smale model of computation implies the set of satisfying assignments of a quantum logic formula to be constructible in the sense of algebraic geometry (Theorem 62a): that is a result in computational complexity serves as a tool to one in expressiveness.

1.3 Geometric versus Abstract Quantum Logic

Quantum Logics of \mathbb{F}^d or of \mathcal{H} are sometimes called *geometric* or *Hilbert Lattices*. *Abstract* quantum logic on the other hand takes the axioms of a modular ortholattice (or of an orthomodular lattice) according to Fact 2 and investigates their consequences; such as the following

Fact 4. a) Let (L, \wedge, \vee, \neg) be an orthomodular lattice and denote by $C(X, Y) := (X \wedge Y) \vee (X \wedge \neg Y) \vee (\neg X \wedge Y) \vee (\neg X \wedge \neg Y)$ denote the (lower) **commutator** of $X, Y \in L$. X, Y are said to **commute** iff $C(X, Y) = 1$; equivalently [Bera85, SECTION 3], if $X = c(X, Y)$ for the (lower) **semi-commutator** $c(X, Y) := (X \wedge Y) \vee (X \wedge \neg Y)$.
b) Now if at least one of X, Y, Z commutes with the other two, then the following distributive laws do apply [Bera85, THEOREM II.3.10]:

$$X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z), \quad X \vee (Y \wedge Z) = (X \vee Y) \wedge (X \vee Z). \quad (2)$$

c) If Y commutes with each of X_1, \dots, X_n , then it also commutes with $f(X_1, \dots, X_n)$ for every formula f [Bera85, THEOREM II.4.4].
d) If X_1, \dots, X_n pairwise commute, they generate a Boolean algebra isomorphic to $\{0, 1\}^k$ for some $k \leq n$ [Bera85, top of p.14 and THEOREM II.4.5].

Concerning Hilbert Lattices, it is known that \mathcal{H} is finite-dimensional iff $\text{Gr}(\mathcal{H})$ satisfies the following modular law:

$$d) \quad P \subseteq R \quad \Rightarrow \quad P \vee (Q \wedge R) = (P \vee Q) \wedge R.$$

On the other hand a theorem of MARIA SOLÉR characterizes which abstract Quantum Logics are isomorphic to $\text{Gr}(\mathcal{H})$ for some Hilbert space \mathcal{H} over either \mathbb{R} , \mathbb{C} , or quaternions \mathbb{H} [Maye98].

Conversely, the so-called orthoarguesian law has been found by ALAN DAY to hold in Hilbert Lattices but not in every abstract Quantum Logic; cf. e.g. [Maye07, Megi09].

1.4 Finite Dimensions: Geometric versus Hilbert Quantum Logic

We mostly restrict to the cases $d = 2$ and $3 \leq d < \infty$, i.e. consider the geometric quantum logics $\text{Gr}(\mathbb{F}^2)$ and $\text{Gr}(\mathbb{F}^d)$. The first corresponds to quantum computing over Qubits $x|0\rangle + y|1\rangle \in \mathbb{F}^2$. Also note that dimensions 2 and 3 are particularly accessible to human intuition. (The evolutionary ‘purpose’ of our brains is to recognize faces and to avoid trees while running through a forest.) In fact the present work shows already these dimensions to exhibit a very rich and interesting structure! We furthermore expect $\text{Gr}(\mathbb{F}^d)$, $d < \infty$, to provide a guiding line for later approaching the infinite-dimensional case; and to open new views to the (exhaustively studied) Boolean case $d = 1$; cmp. e.g. Lemma 30b) below.

2 Quantum Logic Formulas: Truth, Equivalence, and Satisfiability

We start right away with some natural

Definition 5. a) A (quantum logic) formula f over variables X_1, \dots, X_n is a syntactically correct expression over X_i, \neg, \wedge, \vee . We sometimes write $f(\vec{X})$ to emphasize the role of the variables $(X_1, \dots, X_n) =: \vec{X}$.

- b) For \mathcal{L} a modular ortholattice and $Z \in \mathcal{L}$, the interval $[0, Z]$ is the set $\{X \in \mathcal{L} : X \preceq Z\}$ equipped with the operations \vee, \wedge restricted from \mathcal{L} to $[0, Z]$ and with negation^{***} $\neg_Z X := Z \wedge \neg X$.
- c) For closed subspaces $U_1, \dots, U_n \subseteq Z$ of \mathcal{H} , the value in Z of a formula $f(\vec{X})$ is written as $\Xi_Z(f; \vec{U})$ and defined recursively as
- $\Xi_Z(X_i; \vec{U}) := U_i$,
 - $\Xi_Z(g \wedge h; \vec{U}) := \Xi_Z(g; \vec{U}) \wedge \Xi_Z(h; \vec{U})$,
 - $\Xi_Z(g \vee h; \vec{U}) := \Xi_Z(g; \vec{U}) \vee \Xi_Z(h; \vec{U})$, and
 - $\Xi_Z(\neg g; \vec{U}) := \neg_Z \Xi_Z(g; \vec{U})$
- i.e. the latter negation is meant within Z (and in fact the reason for having to supply the index Z to Ξ). Only when Z is clear from the context, we shall simply write $f(\vec{U})$ for $\Xi(f; \vec{U})$ or even re-use variable symbols X_i to denote values from $\text{Gr}(Z)$.
- d) A quantum formula f with arguments $U_1, \dots, U_n \subseteq Z$ evaluates to **true** (i.e. “= 1”) in Z if $\Xi_Z(f; \vec{U}) = Z$ holds; it is **weakly true** it does not evaluate to **false** (i.e. “ $\neq 0$ ”) in Z .
- e) A formula $f(X_1, \dots, X_n)$ is (strongly) **satisfiable** in Z if there exist $U_1, \dots, U_n \in \text{Gr}(Z)$ such that $\Xi_Z(f; \vec{U}) = Z$. It is **weakly satisfiable** by $U_1, \dots, U_n \in \text{Gr}(Z)$ such that $\Xi_Z(f; \vec{U}) \neq 0$.
- f) Two formulas $f(X_1, \dots, X_n)$ and $g(X_1, \dots, X_n)$ are **equivalent** over $\text{Gr}(\mathcal{H})$ if $\Xi_Z(f; \vec{U}) = \Xi_Z(g; \vec{U})$ holds for all $U_1, \dots, U_n \in \text{Gr}(Z)$. They are **weakly equivalent** over $\text{Gr}(Z)$ if it holds

$$\forall U_1, \dots, U_n \in \text{Gr}(Z) : \quad \Xi_Z(f; \vec{U}) = 1 \quad \Leftrightarrow \quad \Xi_Z(g; \vec{U}) = 1 \quad .$$

Note that weak satisfiability of f amounts to the lack of (i.e. in-) validity of the equation “ $f = 0$ ”.

Remark 6. In dimensions > 1 , the connective “ \vee ” behaves like Boolean disjunction for weak satisfiability: $X \vee Y \neq 0$ holds iff $X \neq 0$ or $Y \neq 0$ holds; but “ \wedge ” is different from Boolean conjunction: $X \wedge Y \neq 0$ may well fail for both $X, Y \neq 0$. Dually, “ \wedge ” behaves in a Boolean way for satisfiability but “ \vee ” does not. Furthermore, Boolean negation has to be distinguished from complement: $X \neq 0 \not\stackrel{\text{def}}{\Leftrightarrow} \neg X = 0$.

Whenever in the sequel this distinction needs emphasis, the Boolean connectives shall be denoted as in the C programming language: `||` for “or”, `&&` means “and”, and `!` denotes negation.

We also remark that for $Z \in \mathcal{L}$, the interval $[0, Z]$ constitutes again a modular ortholattice; and for $U_1, \dots, U_n \preceq Z$, $\Xi_Z(\cdot; \vec{U}) : f \mapsto \Xi_Z(f; \vec{U})$ is a homomorphism from the free modular ortholattice on n generators to $[0, Z]$. Its image is called the modular ortholattice *generated* by \vec{U} and denoted by $\langle \vec{U} \rangle_{[0, Z]}$. Moreover, for $Z \in \mathcal{L} := \text{Gr}_k(\mathbb{F}^d)$, $[0, Z]$ is isomorphic to $\text{Gr}(\mathbb{F}^k)$.

Example 7. a) $\neg X \vee \neg Y$ and $\neg(X \wedge Y)$ are formulas over variables X, Y . They are equivalent over any Hilbert space.

b) Suppose $X \subseteq \neg Y$ and $X \vee Y = 1$. Then $X = \neg Y$.

c) Recall the commutator $C(X, Y)$ of X, Y . Whenever $X \in \{0, 1, Y, \neg Y\}$ or $Y \in \{0, 1, X, \neg X\}$, it follows $C(X, Y) = 1$. In particular, $C(X, Y)$ is equivalent to 1 on $\text{Gr}(\mathbb{F}^1)$. Over $Z := \mathbb{F}^2$, however, $C(\text{lspan}(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}), \text{lspan}(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}))$ evaluates to 0.

^{***} Generalizing this binary operation $(X, Z) \mapsto \neg_Z X$, [HMR05, p.144] considers *relative* modular ortholattices.

d) Let $f(X, Y) := C(X, Y) \vee X \vee Y$ and $g(X, Y, Z) := f(X, Y) \wedge f(X, Z) \wedge f(Y, Z)$. Then g is equivalent to 1 on $\text{Gr}(\mathbb{F}^2)$.

Over \mathbb{F}^3 , however, $g(\text{lspan}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \text{lspan}\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), \text{lspan}\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right))$ evaluates to 0.

e) Formulas $f(X, Y) := X \wedge (\neg X \vee Y)$ and $g(X, Y) := Y \wedge (\neg Y \vee X)$ are weakly equivalent, and so are their complements $\neg f$ and $\neg g$. They are, however, not equivalent on $\text{Gr}(\mathbb{F}^2)$.

f) Let $X \in \text{Gr}_1(\mathbb{F}^3)$ and $Y \in \text{Gr}_2(\mathbb{F}^3)$ be a line and a plane, respectively. Then it holds $X \subseteq Y \Leftrightarrow \neg X \vee (X \wedge Y) = 1$ and $X \not\subseteq Y \Leftrightarrow X \vee Y = 1$.

g) For $X, Y \in \text{Gr}(\mathcal{H})$ it holds

$$\begin{aligned} X = Y &\Leftrightarrow (X \wedge Y) \vee (\neg X \wedge \neg Y) = 1 \Leftrightarrow (X \vee Y) \wedge (\neg X \vee \neg Y) = 0 \\ X < Y &\Leftrightarrow X \leq Y \ \&\& \ \neg X \wedge Y > 0. \end{aligned} \quad (3)$$

Finally, in $\text{Gr}(\mathbb{F}^d)$, $X \vee \neg Y = 1$ implies $\dim(Y) \leq \dim(X)$.

Claims a) and b) can be found in any textbook on Hilbert spaces; compare Fact 2. Claim c) is straightforward to verify. Claim d) has been generalized in [Hagg07] to formulas ψ_n equivalent to 1 over \mathbb{F}^n but nonequivalent over \mathbb{F}^m for any $m > n$; see also Lemma 40c) below.

Proof. d) By c), $f(X, Y) = 1$ whenever either of X, Y is 0-dimensional or equal to the (complement of the) other. Also, if X, Y are distinct 1-dimensional subspaces on the plane \mathbb{F}^2 , then $X \vee Y = \mathbb{F}^2$; hence f (and thus g as well) is equivalent to 1 over $\text{Gr}(\mathbb{F}^2)$. However for 1-dimensional subspaces (i.e. lines) $X \notin \{Y, \neg Y\}$ in \mathbb{F}^3 , $C(X, Y) = 0$ hence $f(X, Y) = X \vee Y$; and the three lines above satisfy $(X \vee Y) \wedge (X \vee Z) \wedge (Y \vee Z) = 0$.

e) If $f(X, Y) = 1$, then necessarily $X = 1$ and $Y = 1$; hence $g(X, Y) = 1$. Similarly, $g(X, Y) = 1$ implies $f(X, Y) = 1$: therefore f and g are weakly equivalent. Concerning weak equivalence of their complements, $f(X, Y) = 0$ means $\tilde{X} \vee \tilde{Y} = 1$ for $\tilde{X} := X \wedge \neg Y \subseteq X =: \neg \tilde{Y}$; hence Item b) implies $X = X \wedge \neg Y$ and thus $X \subseteq \neg Y$, i.e. $g(X, Y) = 0$. For $X = \begin{smallmatrix} 1 \\ 0 \end{smallmatrix}$ and $Y = \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$, however, $f(X, Y) = X \neq Y = g(X, Y)$: hence f and g lack equivalence.

f) If $X \subseteq Y$, then $X \wedge Y = X$ hence $\neg X \vee (X \wedge Y) = 1$; also $X \vee Y = Y \subsetneq \mathbb{F}^3 = 1$. If $X \not\subseteq Y$, then $X \wedge Y = 0$ hence $\neg X \vee (X \wedge Y) = \neg X \neq 1$; also $X \vee Y = \mathbb{F}^3$ for reasons of dimension.

g) The implications “ \Rightarrow ” are straight forwardly verified; and also “ \Leftarrow ” is obvious in the second line of Equation (3). The second equivalence in the first line holds by de Morgan’s law (Fact 2). So suppose $0 = (X \vee Y) \wedge \neg(X \wedge Y)$. Then $A := X \wedge Y$ has $A \leq C := X \vee Y$; hence Fact 2 implies with $B := \neg A$:

$$X \wedge Y = A = A \vee \underbrace{(B \wedge C)}_{=0} \stackrel{\text{Eq. (1)}}{=} \underbrace{(A \vee B)}_{=1} \wedge C = C = X \vee Y$$

and thus $X = Y$. For the final claim calculate $d = \dim(X \vee \neg Y) \leq \dim(X) + \dim(\neg Y) = \dim(X) + d - \dim(Y)$. \square

An alternative proof of Claim g) in [DHMW05, p.355 l.7], tailored to the finite-dimensional case, seems to miscalculate $n = \dim(p \vee q) + \dim(\neg p \vee \neg q) - \dim((p \vee q) \wedge (\neg p \vee \neg q)) \neq \dim(p \vee q) + \dim(\neg p \vee \neg q)$.

Remark 8. a) *The Inclusion-Exclusion Principle of finite sets may be suggestive to apply also to the finite dimensions of subspaces, but it does not[†]: $\dim(X \vee Y \vee Z) =$*

$$\dim(X) + \dim(Y) + \dim(Z) - \dim(X \wedge Y) - \dim(X \wedge Z) - \dim(Y \wedge Z) + \dim(X \wedge Y \wedge Z)$$

fails, e.g., for $X := \text{lspan} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $Y := \text{lspan} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $Z := \text{lspan} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

b) *Like in the Boolean case, there are precisely 4 nonequivalent univariate formulas: $0, 1, X, \neg X$. Indeed, any combination of connectives \vee, \wedge, \neg and literals $0, 1, X, \neg X$ simplifies again to one of $0, 1, X, \neg X$ independent of the underlying geometric space.*

c) *By Definition 5f), two formula f and g are equivalent over $\text{Gr}(\mathcal{H})$ iff their values agree on all possible inputs, that is on every choice of subspaces of \mathcal{H} . And, as opposed to the Boolean case $\dim(Z) = 1$, there are infinitely many of them for $\dim(Z) > 1$. Corollary 12c) below identifies finite and small collections of inputs that provably yield different values on 2D nonequivalent formula. By Example 7g) it suffices to consider weak satisfiability, i.e. to look for subspaces $\vec{U} \subseteq \mathcal{H}$ such that $f(\vec{U}) \neq 0$.*

2.1 Generic Arguments and their Values

Nondegeneracy is a common standard hypothesis in computational geometry [GoOR04]. We adopt this concept in the following

Definition 9. *Call a family $0 \neq U_i \in \text{Gr}(\mathcal{H})$, $i \in I$, pairwise generic in \mathcal{H} if it holds*

$$\forall i \neq j : \quad 0 = U_i \cap U_j = U_i \cap \neg U_j = \neg U_i \cap \neg U_j . \quad (4)$$

In case $\text{Card}(I) = 1$, require $0 < U < 1$.

It follows that $U_i \notin \{U_j, \neg U_j, 0, 1\}$ and that $1 = U_i \vee U_j = U_i \vee \neg U_j = \neg U_i \vee \neg U_j$. Put differently, $\{0, 1, U_i, \neg U_i : i \in I\}$ is isomorphic to $\mathcal{MO}_{|I|}$, the free modular ortholattice of height 2 on $2|I|$ atoms and their complements; cf. Figure 1 and Lemma 10c) below. Also, a family $(U_i)_{i \in I}$ is pairwise generic iff $C(U_i, U_j) = 0$ for all $i \neq j$. In particular, every subfamily or permutation of a pairwise generic one is itself pairwise generic.

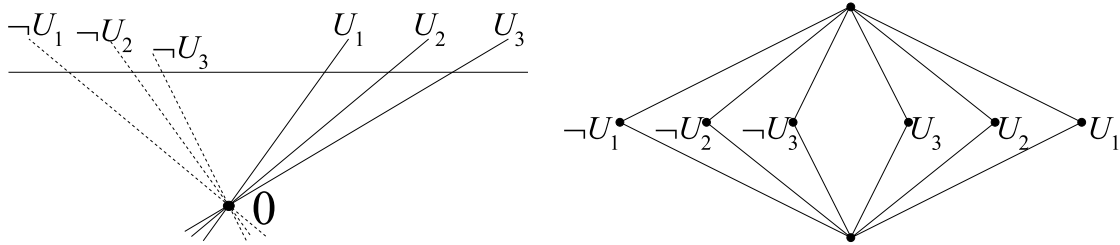


Fig. 1. A family of 3 pairwise generic lines in the plane (left) and Hasse Diagram of the ortholattice \mathcal{MO}_3 (right).

Lemma 10. a) *Let $f(X_1, \dots, X_n)$ be a formula and (U_1, \dots, U_n) pairwise generic.*

Then $f(\vec{U}) \in \{0, 1, U_1, \neg U_1, \dots, U_n, \neg U_n\}$.

[†] Compare also <http://unapologetic.wordpress.com/2008/07/24/the-inclusion-exclusion-principle/>

- b) Let \vec{U} be pairwise generic in \mathcal{H} and \vec{V} pairwise generic in some unitary \mathbb{E} -vector space \mathcal{H}' where $\mathbb{E} \subseteq \mathbb{C}$ denotes a field. Let $f(\vec{X})$ denote a formula.
- If $f(\vec{U}) \in \{0, 1\}$, then $f(\vec{V}) = f(\vec{U})$.
 - If $f(\vec{U}) = U_i$, then $f(\vec{V}) = V_i$.
 - If $f(\vec{U}) = \neg U_i$, then $f(\vec{V}) = \neg V_i$.
- c) Under the hypothesis of b), $U_i \mapsto V_i$ extends to an isomorphism $\langle \vec{U} \rangle_{\text{Gr}(\mathcal{H})} \rightarrow \langle \vec{V} \rangle_{\text{Gr}(\mathcal{H}')}.$
- d) In even dimensions $\mathcal{H} = \mathbb{F}^{2d}$ there exists an explicit infinite pairwise generic family $U_i \in \text{Gr}(\mathcal{H})$, $i \in I$.
- e) Continuing Example 7c), $\neg C(X, Y) = (X \vee Y) \wedge (\neg X \vee Y) \wedge (X \vee \neg Y) \wedge (\neg X \vee \neg Y)$ is satisfiable over each $\text{Gr}(\mathbb{F}^{2d})$, $d \in \mathbb{N}$; but not over $\text{Gr}(\mathbb{F}^{2d-1})$ nor over $\text{Gr}(\mathbb{F}^\infty)$.

The middle part of Claim e) shows that Item d) cannot be extended to odd dimensions.

- Proof.* a) Straightforward induction shows that the set $\{0, 1, U_1, \neg U_1, \dots, U_n, \neg U_n\}$ is closed under \neg, \wedge, \vee , and hence under f .
- b) Fix \vec{U} and apply induction on the length of f . Base cases are $f(\vec{X}) = 0$, $f(\vec{X}) = 1$, and $f(\vec{X}) = X_i$. In the induction step, we verify the cases i) $f = \neg g$, ii) $f = g \wedge h$, and iii) $f = g \vee h$ where g and h satisfy the claim by induction hypothesis.
- i) If $f(\vec{U}) \in \{0, 1\}$ then $g(\vec{U}) \in \{0, 1\}$, so $g(\vec{V}) = g(\vec{U})$, hence $f(\vec{V}) = f(\vec{U})$.
 If $f(\vec{U}) = U_i$ then $g(\vec{U}) = \neg U_i$, so $g(\vec{V}) = \neg V_i$, hence $f(\vec{V}) = V_i$.
 Similarly for $f(\vec{U}) = \neg U_i$.
- ii) If $g(\vec{U}) = 0$ then $g(\vec{V}) = 0$, hence $f(\vec{V}) = g(\vec{V}) \wedge h(\vec{V}) = 0 = g(\vec{U}) \wedge h(\vec{U}) = f(\vec{U})$.
 Analogously for $h(\vec{U}) = 0$. And if $g(\vec{U}) = 1 = h(\vec{U})$, then $f(\vec{V}) = 1 = f(\vec{U})$.
 If $g(\vec{U}) = 1$ and $h(\vec{U}) = U_i$, then $f(\vec{U}) = U_i$ and $f(\vec{V}) = V_i$. Similarly for $g(\vec{U}) = 1$ and $h(\vec{U}) = \neg U_i$; and analogously for $g(\vec{U}) \in \{U_i, \neg U_i\}$, $h(\vec{U}) = 1$.
 The case $g(\vec{U}) = h(\vec{U})$ is obvious as well.
 Now consider $g(\vec{U}) = U_i$ and $h(\vec{U}) = U_j$ with $i \neq j$. Then, by Equation (4), $f(\vec{U}) = U_i \wedge U_j = 0 = V_i \wedge V_j = f(\vec{V})$. Similarly for $g(\vec{U}) = \neg U_i$ and/or $h(\vec{U}) = \neg U_j$.
- iii) analogous.
- c) Map $f(U_1, \dots, U_n) \in \langle \vec{U} \rangle$ to $f(V_1, \dots, V_n) \in \langle \vec{V} \rangle$. This is well-defined according to b). It extends $U_i \mapsto V_i$ and respects \vee, \wedge, \neg : hence is a homomorphism. Bijectivity follows by symmetry.
- d) Define $I := (0, \infty) \cap \mathbb{Q} \subseteq \mathbb{F}$. First consider the planar case $\mathcal{H} = \mathbb{F}^2$ by letting $U_i := \text{lspan}(\begin{smallmatrix} 1 \\ i \end{smallmatrix})$; hence $\neg U_i = \text{lspan}(\begin{smallmatrix} 1 \\ -1/i \end{smallmatrix})$, and Equation (4) is readily verified.
 This construction extends to other even dimensions as follows: Decompose $\mathcal{H} = Z \times Z$ and let $U_i := \{(\vec{x}, i\vec{x}) : \vec{x} \in Z\}$, $i \in I$.
- e) We have already remarked that a pairwise generic tuple (X, Y) has $C(X, Y) = 0$. Conversely suppose $\neg C(X, Y) = 1$ for some $X, Y \in \text{Gr}(\mathbb{F}^{2d-1})$. Then $1 = X \vee Y = \neg X \vee Y = X \vee \neg Y = \neg X \vee \neg Y$ requires $\dim(X) = \dim(Y)$ and $2d - 1 = \dim(X) + \dim(Y)$: contradiction. Finally consider the case $\neg C(X, Y) = \mathbb{F}^\infty$ for $X, Y \in \text{Gr}(\mathbb{F}^\infty)$. Then $\infty = \dim(X \vee Y) \leq \dim(X) + \dim(Y)$ and similarly $\infty \leq \dim(\neg X) + \dim(Y)$, $\infty \leq \dim(X) + \dim(\neg Y)$, $\infty \leq \dim(\neg X) + \dim(\neg Y)$: hence X, Y cannot both have co-/finite dimensions: again a contradiction. \square

Definition 11. For a family $U_i \in \text{Gr}(\mathcal{H})$, $i \in I$, its degree of pairwise genericity is the cardinality of a largest pairwise generic subfamily:

$$\deg\{U_i : i \in I\} := \max \left\{ \text{Card}(J) : J \subseteq I, (U_i)_{i \in J} \text{ pairwise generic} \right\}$$

In 2D, the sub-ortholattice generated by $(U_i)_{i \in I}$ turns out as isomorphic to \mathcal{MO}_{2d} where $d := \text{Card}(U_1, \dots, U_n)$. In particular, it depends only on d ; and \mathcal{MO}_{2d} can be embedded into $\mathcal{MO}_{2d'}$ whenever $d \leq d'$:

Corollary 12. *Fix some family $U_i \in \text{Gr}(\mathbb{F}^2), i \in I$; and let $J \subseteq I$ be of $\text{Card}(J) = \deg(U_i : i \in I)$ such that $(U_i)_{i \in J}$ is pairwise generic.*

- a) *For all $i \in I$, it holds $U_i \in \{0, 1, U_j, \neg U_j : j \in J\}$.*
- b) *Let $\mathbb{E} \subseteq \mathbb{C}$ denote a field and suppose $V_j \in \text{Gr}(\mathbb{E}^2), j \in J$, is a pairwise generic family. Then $V_j \mapsto U_j$ extends to an isomorphism $\langle V_j : j \in J \rangle_{\text{Gr}(\mathbb{E}^2)} \rightarrow \langle U_i : i \in I \rangle_{\text{Gr}(\mathbb{F}^2)}$.*
- c) *In particular if $\text{Card}(J) = d$ and formula $f(X_1, \dots, X_n)$ is [weakly] satisfied by U_1, \dots, U_n , then there also exist $X_1, \dots, X_n \in \{0, 1, V_1, \neg V_1, \dots, V_d, \neg V_d\}$ [weakly] satisfying f .*
- d) *Slightly more generally suppose $U_i, V_j, W_\ell \in \text{Gr}(\mathbb{F}^2)$ have $\deg(W_1, \dots, W_k, V_1, \dots, V_n) \leq \deg(W_1, \dots, W_k, U_1, \dots, U_m)$. Then there is an injective homomorphism from $\langle \vec{W}, \vec{V} \rangle$ to $\langle \vec{W}, \vec{U} \rangle$ fixing $\langle \vec{W} \rangle$.*
In particular f [weakly] satisfiable over $\langle \vec{W}, \vec{V} \rangle$ is also [weakly] satisfiable over $\langle \vec{W}, \vec{U} \rangle$.

Proof. a) Since $(U_i)_{i \in J}$ is pairwise generic, $0 = \{0\} \subsetneq U_j \subsetneq \mathbb{F}^2 = 1$; hence $\dim(U_j) = 1$. Now suppose there is some $U_i \notin \{0, 1, U_j, \neg U_j : j \in J\}$. Then U_i must have dimension 1, too, i.e. is a line. So, for each $j \in J$, $U_i \neq U_j$ implies that the two lines are linearly independent; hence have zero intersection $U_i \wedge U_j = \{0\}$ and $U_i + U_j = \mathbb{F}^2$. Similarly for $U_i \neq \neg U_j$. This demonstrates that the extended family $(U_j)_{j \in J \sqcup \{i\}}$ is still pairwise generic—contradicting the maximality of J .

- b) By Lemma 10c), $V_j \mapsto U_j$ extends to an isomorphism $\langle V_j : j \in J \rangle_{\text{Gr}(\mathbb{E}^2)} \rightarrow \langle U_j : j \in J \rangle_{\text{Gr}(\mathbb{F}^2)}$; but the latter ortholattice coincides with $\langle U_i : i \in I \rangle_{\text{Gr}(\mathbb{F}^2)}$ by a).
- c) Let $\Psi : \langle U_i : i \in I \rangle \rightarrow \langle V_j : j \in J \rangle = \{0, 1, V_1, \neg V_1, \dots, V_d, \neg V_d\}$ denote the inverse of the isomorphism according to b). Then $X_i := \Psi(U_i)$ has the desired property.
- d) W.l.o.g. suppose \vec{W} is pairwise generic: otherwise drop those $W_i \in \{0, 1, W_j, \neg W_j : j \neq i\}$ according to a). Similarly suppose both (\vec{W}, \vec{V}) and (\vec{W}, \vec{U}) are pairwise generic. Then $n \leq m$; w.l.o.g. $n = m$. By Lemma 10c), $W_i \mapsto W_i, V_j \mapsto U_j$ extends to an isomorphism; which fixes $\langle \vec{W} \rangle$. The second claim follows as in c). \square

Example 13. a) *For $V, W \in \{0, 1, U_1, \dots, U_n, \neg U_1, \dots, \neg U_n\}$, $C(V, W) = 1$ iff $V \in \{0, 1, W, \neg W\}$ or $W \in \{0, 1, V, \neg V\}$ and $C(V, W) = 0$ otherwise.*
b) *Let $f(X_1, \dots, X_n)$ be a quantum formula such that $f(\vec{U}) \neq 0$ for some $\vec{U} \in \text{Gr}(\mathbb{F}^2)$ of degree of pairwise genericity $\deg(\vec{U}) = 1$. Then $f(\vec{W}) = 1$ for some $\vec{W} \in \{0, 1\}^n$.*

Proof. a) The cases $V \in \{0, 1, W, \neg W\}$ or $W \in \{0, 1, V, \neg V\}$ are handled in Example 7c) and Fact 4. The remaining cases $(V, W) = (U_i, U_j), (\neg U_i, U_j), (U_i, \neg U_j), (\neg U_i, \neg U_j), i \neq j$, follow from Definition 9.

- b) According to Corollary 12c), there exist $W_1, \dots, W_n \in \{0, 1, V, \neg V\}$ such that $0 \neq f(\vec{W}) = g(V)$ for a univariate formula g and V some arbitrary 1D (hence trivially pairwise generic) subspaces of \mathbb{F}^2 . Indeed, Lemma 10b) shows that the choice of V does not matter. Now recall Remark 8b) that, up to equivalence, there are exactly 4 such formulas: $g(X) = 0$, $g(X) = 1$, $g(X) = X$, $g(X) = \neg X$. The first case does not apply since $0 \neq g(V)$ by presumption. And in the other three cases, there obviously exists some $V' \in \{0, 1\}$ with $1 = g(V')$. \square

2.2 Number of Nonequivalent Formulas

In the Boolean case $d = 1$, there are exactly 2^{2^n} nonequivalent formulas in n variables: each $f(X_1, \dots, X_n)$ uniquely gives rise to a function $F : \{0, 1\}^n \rightarrow \{0, 1\}$; and conversely any such function can be expressed as the formula

$$f(X_1, \dots, X_n) := \bigvee_{\substack{x_1, \dots, x_n \in \{0, 1\} \\ F(\vec{x})=1}} \bigwedge_{i=1}^n \varphi_{x_i}(X_i), \quad \varphi_1(X) := X, \quad \varphi_0(X) := \neg X. \quad (5)$$

Theorem 18 below shows that in 2D there are asymptotically precisely $2^{2^{\theta(n \cdot \log n)}}$ nonequivalent formulas in n variables.

Note (Remark 8b) that already in case $d > 1 = n$, not all (infinitely many) functions $F : \text{Gr}(\mathbb{F}^d)^n \rightarrow \text{Gr}(\mathbb{F}^d)$ arise from a formula:

Remark 14. *In one variable, and independently of d and of \mathbb{F} of characteristic 0, there are precisely 4 nonequivalent formulas, independent of the dimension d and also in the free ortholattice: namely 0, 1, X , and $\neg X$.*

For $d > 1$, also the number of formulas in two variables independent of d and \mathbb{F} .

Namely the free modular ortholattice generated by two elements is isomorphic to the direct product $\mathcal{MO}_4 \times \{0, 1\}^4$ [Kalm83, THEOREM I.3.9]; and the latter admits an embedding into $\text{Gr}(\mathbb{F}^2)$. The 96 formulas are listed e.g. in [Bera85, FIG.18].

In 3D, the number of nonequivalent three-variate formulas already is infinite. A proof can be found, e.g., in [Bera85, EXAMPLE III.2.15]. We include a graphic visualization of the underlying construction:

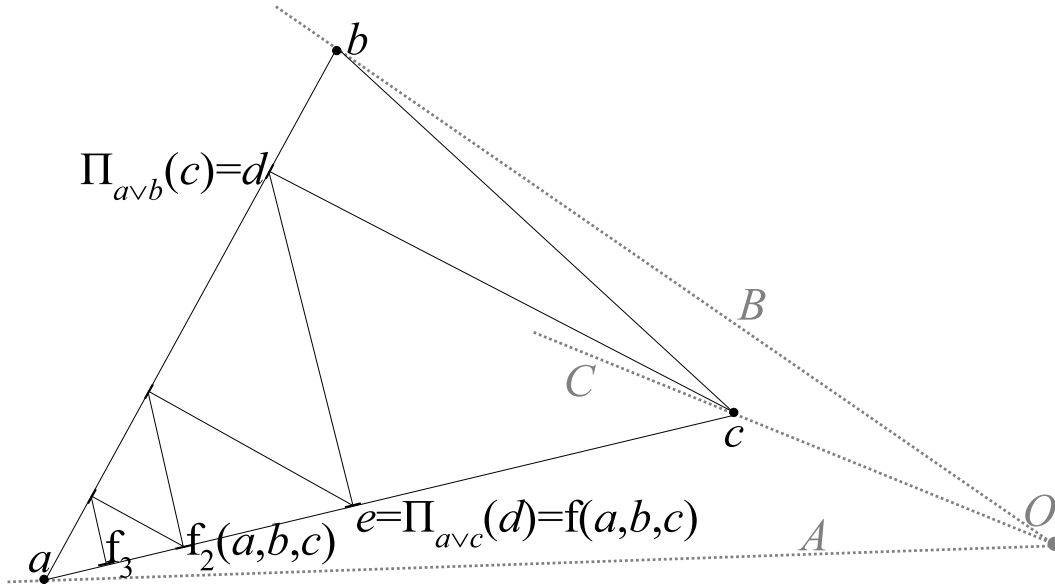


Fig. 2. Illustration of the iterated projection formulas f_n from Example 15. The lines A, B, C through origin O are indicated by the respective vertices a, b, c of their intersections with some skew 2D plane.

Example 15. Consider the (linear, self-adjoint, idempotent) projection $\Pi(\cdot; Z) : \mathcal{H} \rightarrow Z$ onto some arbitrary but fixed closed subspace Z of \mathcal{H} . More abstractly consider the mapping $\Pi(\cdot, Z) : \text{Gr}(\mathcal{H}) \rightarrow \text{Gr}(Z)$ induced by the formula $\Pi(X, Z) = X \mapsto Z \wedge (X \vee \neg Z)$. Now consider the formula

$$f(X, Y, Z) := \Pi(X \vee Z, \Pi(X \vee Y, Z))$$

illustrated in Figure 15. This figure also reveals that there exist lines A, B, C in $\text{Gr}(\mathbb{F}^3)$ such that the iterated sequence of formulas $f_{n+1}(X, Y, Z) := f(X, Y, f_n(X, Y, Z))$ attains values $f_n(A, B, C)$ strictly approaching A . Therefore f_n and f_m are nonequivalent in 3D for $n \neq m$.

The deeper reason underlying Example 15 is that every non-2-distributive [MaRo87] modular sublattice L of $\text{Gr}(\mathbb{F}^d)$ contains, and admits the definition of, a 3-frame spanning entire $\text{Gr}(\mathbb{Q}^3)$; cmp., e.g., [Herr82, HaSv96] and Fact 48 below.

So the sensible question to ask is for the number of formula in n variables nonequivalent over $\text{Gr}(\mathbb{F}^2)$. In view of Lemma 10c), this is independent of \mathbb{F} (of characteristic 0): we may simply speak of 2D non/equivalence. Moreover by Lemma 10c), the modular ortholattice \mathcal{F}_n of 2D nonequivalent formulas in n variables is isomorphic to the n -generated free algebra $F_{\mathcal{MO}_{2n}}(n)$ in the variety \mathcal{MO}_{2n} of $2n$ -element modular ortholattices of height 2. (2D essentially means 2-distributive.) And $F_{\mathcal{MO}_k}(n)$ has been determined in [HKW97, THEOREM 3.3]:

Fact 16. Let (V_1, \dots, V_n) be pairwise generic in $\text{Gr}(\mathbb{F}^2)$. Then \mathcal{F}_n is isomorphic to the direct product

$$2^{2^n} \times \prod_{p=2}^n \{0, 1, V_1, \dots, V_n, \neg V_1, \dots, \neg V_n\}^{\phi(n,p)}, \quad \text{where} \quad \phi(n,p) := 2^{n-p} \sum_{\ell=0}^{n-p} \binom{n}{\ell} S(n-\ell, p)$$

and $S(m, p) = p \cdot S(m-1, p) + S(m-1, p-1) = \frac{1}{p!} \sum_{s=1}^p (-1)^{p-s} \binom{p}{s} s^m$ denote the Stirling numbers of the second kind.

For instance in the case $n = 3$, one obtains $\phi(3, 2) = 12$ and $\phi(3, 3) = 1$ and thus the following

Example 17. Let (A, B, C) be pairwise generic in $\text{Gr}(\mathbb{F}^2)$ and denote by $\mathcal{G}_3(A, B) \subseteq \text{Gr}(\mathbb{F}^2)$ the set of the following 12 triples of degree 2 of pairwise genericity:

$$(A, B, 0), (A, B, 1), (A, B, A), (A, B, \neg A), (A, B, B), (A, B, \neg B), \\ (A, 0, B), (A, 1, B), (A, A, B), (A, \neg A, B), (0, A, B), (1, A, B) .$$

Then the following mapping is a bijection:

$$\begin{aligned} \phi_3 : \mathcal{F}_3 &\rightarrow \{0, 1\}^{\{0, 1\}^3} \times \{0, 1, A, B, \neg A, \neg B\}^{\mathcal{G}_3(A, B)} \times \{0, 1, A, B, C, \neg A, \neg B, \neg C\} \\ \phi_3 : f &\mapsto \left(\left(\{0, 1\}^3 \ni \vec{x} \mapsto f(\vec{x}) \right), \left(\mathcal{G}_3(A, B) \ni \vec{W} \mapsto f(\vec{W}) \right), f(A, B, C) \right) \end{aligned}$$

(As usual, Y^X denotes the set of mapping $f : X \rightarrow Y$.) In particular there are precisely $2^8 \times 6^{12} \times 8$ 2D nonequivalent formulas in three variables.

This number is obviously much larger than 2^{2^3} , the number of nonequivalent Boolean (i.e. 1D) formulas in three variables. We now show that, also asymptotically, the number of nonequivalent 2D formulas in n variables is much larger than in the 1D case. This is possible, but quite technical, to conclude from Fact 16. Instead, we directly deduce simple upper and lower bounds of asymptotically matching second exponents.

- Theorem 18.** a) The number $\text{Card}(\mathcal{F}_n)$ of 2D nonequivalent formulas in n variables is asymptotically bounded from above by $(2n+2)^{(2n+2)^n} \leq 2^{2^{\mathcal{O}(n \log n)}}$.
- b) Let $n \in \mathbb{N}$ and fix some n -tuple (V_1, \dots, V_n) pairwise generic in $\text{Gr}(\mathbb{F}^2)$. Consider any map $F : \{1, \dots, n\}^n \rightarrow \{0, 1\}$. Then there exists a $2n$ -variate formula f such that $f(V_1, \dots, V_n, V_{k_1}, \dots, V_{k_n}) = F(k_1, \dots, k_n)$ for all $1 \leq k_1, \dots, k_n \leq n$.
- c) In particular, there is an injective mapping from $\{0, 1\}^{\{1, \dots, n\}^n}$ to the set \mathcal{F}_{2n} of 2D nonequivalent formulas in $2n$ variables; hence $\text{Card}(\mathcal{F}_{2n}) \geq 2^{n^n} = 2^{2^{n \cdot \log_2 n}}$.

Thus the number of nonequivalent 2D formulas in n variables is $2^{2^{\theta(n \log n)}}$: as opposed to 2^{2^n} in the 1D case.

Proof (Theorem 18).

- a) Consider the set \mathcal{F}_n of 2D nonequivalent formulas in n variables and the mapping

$$\begin{aligned} \phi_n : \mathcal{F}_n &\rightarrow \left(\{0, 1, V_1, \neg V_1, \dots, V_n, \neg V_n\}^n \right)^{\{0, 1, V_1, \neg V_1, \dots, V_n, \neg V_n\}}, \\ \phi_n : F(X_1, \dots, X_n) &\mapsto \left((W_1, \dots, W_n) \mapsto f(W_1, \dots, W_n) \right) \end{aligned}$$

for fixed pairwise generic $(V_1, \dots, V_n) \in \text{Gr}(\mathbb{F}^2)$. By Lemma 10 and Corollary 12, this mapping is well-defined and injective. Hence

$$\text{Card}(\mathcal{F}_n) \leq \text{Card range}(\phi_n) \leq (2n+2)^{(2n+2)^n} = 2^{(2n+2)^n \cdot \log_2(2n+2)}$$

$$\text{and } (2n+2)^n \cdot \log_2(2n+2) = 2^{n \cdot \log_2(2n+2) + \log \log_2(2n+2)} \leq 2^{\mathcal{O}(n \cdot \log n)}.$$

- b) Let

$$f(X_1, \dots, X_n, Y_1, \dots, Y_n) := \bigvee_{\substack{1 \leq k_1, \dots, k_n \leq n \\ F(k_1, \dots, k_n) = 1}} \bigwedge_{i=1}^n C(Y_i, X_{k_i}).$$

Observe that $C(V_j, V_i) = 1$ for $i = j$ and $C(V_j, V_i) = 0$ otherwise, cf. Example 13a). Therefore, similar to Equation (5), $f(V_1, \dots, V_n, V_{\ell_1}, \dots, V_{\ell_n}) = 1$ iff $(\ell_1, \dots, \ell_n) = (k_1, \dots, k_n)$ for $F(k_1, \dots, k_n) = 1$, and $f(V_1, \dots, V_n, V_{\ell_1}, \dots, V_{\ell_n}) = 0$ otherwise.

- c) follows from b). \square

2.3 Formulas with Constants and 2D Quantifier Elimination

Over Booleans, the operations \neg, \vee, \wedge generate the entire logic (and in fact one of \vee, \wedge even disposable) in the sense that any function $f : \{0, 1\}^d \rightarrow \{0, 1\}$ can be expressed by a formula over, say, $\{\neg, \vee\}$. For instance, inequality

$$\psi_{\neq} : \text{Gr}(\mathbb{F}^d) \ni (x, y) \mapsto 1 \quad \text{for } x \neq y, \quad \psi_{\neq}(x, y) = 0 \quad \text{for } x = y \quad (6)$$

is equivalent to $g(x, y) := \neg(x \wedge y) \wedge \neg(\neg x \wedge \neg y)$ in the Boolean case $d = 1$; but their complements are only weakly equivalent in case $d \geq 2$, recall Example 7g) and also Remark 6. Proposition 19 below shows that ψ_{\neq} is in fact no formula for $d = 2$.

Related negative results regarding the expressive power of equality have been obtained in [MePa03]; see also the references therein and, concerning the (model-theoretic) foundations of quantum logic, [Gold84].

Proposition 19. a) On $\dim(\mathcal{H}) \geq 2$, there is no quantum logic formula $f(X)$ such that

$$\psi_{\neq}(0) = 0, \quad \psi_{\neq}(x) = 1 \quad \text{for all } 0 \neq x \in \text{Gr}(\mathcal{H}) \quad (7)$$

b) In 2D there is, however, a three-variate formula $g(X, Y, Z)$ such that for any fixed pairwise generic tuple $(y, z) \in \text{Gr}(\mathbb{F}^2)^2$, the function $f := g(\cdot, y, z) : \text{Gr}(\mathbb{F}^2) \ni x \mapsto g(x, y, z) \in \{0, 1\}$ is well-defined and satisfies Equation (7).

c) On the other hand, there is no bivariate formula $h(X, Y)$ and $y \in \text{Gr}(\mathbb{F}^2)$ such that $f := h(\cdot, y)$ satisfies Equation (7).

The function ψ_{\neq} according to Equation (7) is known as *indiscrete quantifier*; cmp. e.g. [Roma06]. Such a function adds considerably to the expressiveness of quantum logic, for instance in view of Remark 6:

Example 20. Let ψ_{\neq} satisfy Equation (7). Then it holds $\psi_{\neq}(X) = 1 \Leftrightarrow \psi_{\neq}(X) \neq 0$. In particular, composition with ψ_{\neq} makes quantum connectives behave like the Boolean ones:

$$\begin{aligned} \psi_{\neq}(X) \vee \psi_{\neq}(Y) = 1 &\iff (X = 1) \parallel (Y = 1) \\ \psi_{\neq}(X) \wedge \psi_{\neq}(Y) \neq 0 &\iff (X \neq 0) \&\& (Y \neq 0) \\ \psi_{\neq}(X) \neq 0 &\iff \neg \psi_{\neq}(X) = 0 . \end{aligned}$$

Notice how Proposition 19b) considers formulas in which some dedicated variables are replaced by fixed values from $\text{Gr}(\mathcal{H})$. This suggests the following extension of Definition 5.

Definition 21. A formula in variables X_1, \dots, X_n with constants $C_1, \dots, C_m \in \text{Gr}(Z)$ is a formula f in $n + m$ variables where $(X_{n+1}, \dots, X_{n+m})$ are fixed to (C_1, \dots, C_m) . Its value at $U_1, \dots, U_n \subseteq Z$ is defined as above.

Proposition 19 may thus be rephrased: The 2D indiscrete quantifier can be expressed as a quantum logic formula with two, but not with less, constants.

Proof (Proposition 19).

- a) Obvious since $0, 1, X, \neg X$ are the only univariate formulas.
- b) Let $g(X, Y, Z) := \neg(C(X, Y) \wedge C(X, Z) \wedge \neg X)$. Then obviously $g(0, y, z) = 0$ and $g(1, y, z) = 1$. Moreover any $x \in \text{Gr}_1(\mathbb{F}^2)$ fails commutativity with at least one y, z because the latter are pairwise generic; hence $C(x, y) \wedge C(x, z) = 0$ and $g(x, y, z) = 1$ in this case.
- c) Suppose the converse. Then a) requires $y \notin \{0, 1\}$; hence $y \in \text{Gr}_1(\mathbb{F}^2)$. From the 96 Beran formulas f_i [Bera85, FIG.18], exactly the following $1 \leq i \leq 96$ satisfy $f_i(0, y) = 0$: 1-3, 6; 17-19, 22; 33-35, 38; 49-51, 54; 65-67, 70; 81-83, 86. And they all have $f_i(y, y) < 1$. \square

Boolean formulas are easily seen to admit quantifier elimination:

Fact 22. For a formula $f(X; Y_1, \dots, Y_n)$, consider the formula $g(\vec{Y}) := f(0; \vec{Y}) \vee f(1; \vec{Y})$. Then, for any choice of $Y_1, \dots, Y_n \in \{0, 1\}$, it holds

$$\exists X \in \{0, 1\} : f(X; \vec{Y}) = 1 \iff g(\vec{Y}) = 1 .$$

We generalize this to the case of 2D quantum logic formulas with constants with respect to both truth and weak truth (Definition 5d).

Theorem 23. Let $f(c_1, \dots, c_m; X, Y_1, \dots, Y_n)$ be a formula with m constants in $n + 1$ variables; recall the degree of pairwise genericity from Definition 11.

- a) There exist $U_1, \dots, U_{n+1} \in \text{Gr}_1(\mathbb{F}^2)$ such that $\deg(c_1, \dots, c_m, U_1, \dots, U_{n+1}) \geq \deg(c_1, \dots, c_m) + n + 1$.
- b) Let $U_1, \dots, U_N \in \text{Gr}_1(\mathbb{F}^2)$ such that $\deg(c_1, \dots, c_m, U_1, \dots, U_N) \geq \deg(c_1, \dots, c_m) + n + 1$. Define formula $g_f(\vec{c}, \vec{U}; \vec{Y})$ in n -variables and with $m + N$ constants as $f(\vec{c}; 0, \vec{Y}) \vee$

$$\vee f(\vec{c}; 1, \vec{Y}) \vee \bigvee_{i=1}^m f(\vec{c}; c_i, \vec{Y}) \vee \bigvee_{i=1}^m f(\vec{c}; -c_i, \vec{Y}) \vee \bigvee_{i=1}^n f(\vec{c}; Y_i, \vec{Y}) \vee \bigvee_{i=1}^n f(\vec{c}; -Y_i, \vec{Y}) \vee \bigvee_{i=1}^N f(\vec{c}; U_i, \vec{Y}).$$

Then, for every choice of $Y_1, \dots, Y_n \in \text{Gr}(\mathbb{F}^2)$, it holds

$$\exists X \in \text{Gr}(\mathbb{F}^2) : f(\vec{c}; X, \vec{Y}) \neq 0 \quad \Leftrightarrow \quad g_f(\vec{c}, \vec{U}; \vec{Y}) \neq 0 .$$

- c) Recall the formula ψ_{\neq} with two constants from Proposition 19b) and let $\tilde{f} := \neg\psi_{\neq}(\neg f)$. Then $g_{\tilde{f}}$ is a formula with at most $m + N + 2$ constants satisfying

$$\exists X \in \text{Gr}(\mathbb{F}^2) : f(X, \vec{Y}) = 1 \quad \Leftrightarrow \quad g_{\tilde{f}}(\vec{Y}) = 1 .$$

- d) Let $f(\vec{X}, \vec{Y}, \vec{Z})$ be a formula without constants. Then there exists a formula $g(\vec{Z}, \vec{U})$, again without constants, such that it holds for all $\vec{Z} \in \text{Gr}(\mathbb{F}^2)$:

$$\forall \vec{X} \in \text{Gr}(\mathbb{F}^2) \exists \vec{Y} \in \text{Gr}(\mathbb{F}^2) : f(\vec{X}, \vec{Y}, \vec{Z}) \stackrel{=}{\neq} 1 \quad \Leftrightarrow \quad \exists \vec{U} \in \text{Gr}(\mathbb{F}^2) : g(\vec{Z}, \vec{U}) \stackrel{=}{\neq} 1 .$$

Note that the introduction of new constants in Theorem 23b+c) cannot in general be avoided:

Example 24. a) “ $\exists X : \neg C(X, Y) = 1$ ” holds over $\text{Gr}(\mathbb{F}^2)$ precisely for $\dim(Y) = 1$; as does “ $\exists X : \neg C(X, Y) \neq 0$ ”.

- b) But there is no univariate formula $g(Y)$ evaluating over $\text{Gr}(\mathbb{F}^2)$ to **true** (non-**false**) exactly for 1D arguments.

- c) In Proposition 19b), one may replace the formula with constants by quantified formulas:

$$\begin{aligned} Y = 1 & \Leftrightarrow \exists X_1, X_2 : Y \wedge C(X_1, Y) \wedge C(X_2, Y) \wedge \neg C(X_1, X_2) \neq 0 \\ & \Leftrightarrow \forall X_1, X_2 : C(X_1, X_2) \vee (Y \wedge C(X_1, Y) \wedge C(X_2, Y)) \neq 0 \\ Y \neq 0 & \Leftrightarrow \forall X_1, X_2 : \neg(Y \wedge C(X_1, Y) \wedge C(X_2, Y) \wedge \neg C(X_1, X_2)) = 1 \\ & \Leftrightarrow \exists X_1, X_2 : \neg C(X_1, X_2) \wedge \neg(Y \wedge C(X_1, Y) \wedge C(X_2, Y)) = 1 \end{aligned}$$

Proof (Example 24c). Note that $C(V, W) \in \{0, 1\}$ for all $V, W \in \text{Gr}(\mathbb{F}^2)$. Now in the first equivalence, pairwise generic (X_1, X_2) yield $\neg C(X_1, X_2) = 1$; and $Y \wedge C(X_1, Y) \wedge C(X_2, Y)$ evaluates to 1 in case $Y = 1$; whereas in case $Y = 0$ it always evaluates to 0; and in case $0 < Y < 1$, $C(X_i, Y) \neq 0$ requires $X_i \in \{0, 1, Y, -Y\}$, hence $C(X_1, X_2) = 1$.

In the second equivalence similarly, $Y = 1$ renders $C(X_i, Y)$ for all X_i ; whereas for $0 < Y < 1$, pairwise generic Y, X_1, X_2 yield $C(X_1, X_2) = 0$ and $C(X_i, Y) = 0$; similarly for $Y = 0$.

The third equivalence is just the contraposition of the first, negated and applied to $\neg Y$; similarly for the fourth and second. \square

Also observe that $\forall X : f(X, \vec{Y}) = 1$ holds iff $\exists X : \neg f(X, \vec{Y}) \neq 0$ fails, i.e. iff $g_{\neg f}(\vec{Y}) = 0$ according to Theorem 23b); similarly for “ $\forall X : f(X, \vec{Y}) \neq 0$ ”. Hence iterative application of Theorem 23 yields

Corollary 25. *a) 2D quantum logic with constants admits quantifier elimination:*

To any formula $f(\vec{c}; \vec{Y})$ with constants $\vec{c} \in \text{Gr}(\mathbb{F}^2)$ and parameters \vec{Y} , first-order quantified over (thus bound) variables $\vec{X} \in \text{Gr}(\mathbb{F}^2)$, there exists a quantifier-free formula $g(\vec{c}', \vec{Y})$ with constants $\vec{c}' \in \text{Gr}(\mathbb{F}^2)$ weakly equivalent to $f(\vec{c}; \vec{Y})$.

b) 2D quantum logic without constants is model complete:

To any formula $f(\vec{Y})$ with parameters \vec{Y} , first-order quantified over (thus bound) variables $\vec{X} \in \text{Gr}(\mathbb{F}^2)$, there exists a purely existentially quantified constant-free formula $g(\vec{Y})$ weakly equivalent to $f(\vec{Y})$.

Theorem 62 below will show that quantum logic with constants does not admit quantifier elimination in dimension 3 and beyond but is model-complete.

Proof (Theorem 23).

- a) Let $1 \leq i_1 < \dots < i_d \leq m$ such that $d = \deg(c_1, \dots, c_m) = \deg(c_{i_1}, \dots, c_{i_d})$. Now consider $U_1, \dots, U_{2d+n+1} \in \text{Gr}(\mathbb{F}^2)$ pairwise generic according Lemma 10d). Note that each U_i may coincide with at most one of $c_{i_1}, \dots, c_{i_d}, \neg c_{i_1}, \dots, \neg c_{i_d}$; hence at least $n+1$ among U_1, \dots, U_{2d+n+1} do not occur in, i.e. remain pairwise generic when extending, $\{c_{i_1}, \dots, c_{i_d}, \neg c_{i_1}, \dots, \neg c_{i_d}\}$.
- b) If $g_f(\vec{c}, \vec{U}; \vec{Y}) > 0$ then at least one of its terms must be > 0 . These are of the form $f(\vec{c}; X, \vec{Y})$ for $X \in \text{Gr}(\mathbb{F}^2)$, hence $\exists X : f(\vec{c}; X, \vec{Y}) > 0$. Conversely let $f(\vec{c}; X, \vec{Y}) \neq 0$ for some $X \in \text{Gr}(\mathbb{F}^2)$. We have $\deg(c_1, \dots, c_m, X, Y_1, \dots, Y_n) \leq \deg(\vec{c}) + n + 1 \leq \deg(\vec{c}) + \deg(\vec{U}) \leq \deg(\vec{c}) + \deg(\vec{Y}, \vec{U})$. So, according to Corollary 12d), there is an injective homomorphism from $\langle \vec{c}, \vec{Y}, X \rangle$ to $\langle \vec{c}, \vec{Y}, \vec{U} \rangle$ fixing $\langle \vec{c}, \vec{Y} \rangle$ and there exists $X' \in \{0, 1, c_1, \dots, c_m, \neg c_1, \dots, \neg c_m, Y_1, \dots, Y_n, \neg Y_1, \dots, \neg Y_n, U_1, \dots, U_N\}$ with $f(\vec{c}; X', \vec{Y}) \neq 0$ occurring as one of the terms of g_f .
- c) Omitting the constants, note that $\tilde{f}(X, \vec{Y}) \in \{0, 1\}$ and $f(X, \vec{Y}) = 1 \Leftrightarrow \tilde{f}(X, \vec{Y}) = 1$ holds for every $X, \vec{Y} \in \text{Gr}(\mathbb{F}^2)$. Hence the disjunction in the definition of $g_{\tilde{f}}$ behaves like Booleans in the sense that it evaluates to 1 iff at least one of its terms does; cmp. Example 20.
- d) Observe that the constants introduced in the proofs of b+c) are pairwise generic tuples \vec{U} ; and it does not actually matter *which* such tuple, as long as its degree of pairwise genericity is large enough. So instead of implementing \vec{U} as constants, we existentially quantify over \vec{U} and require it to be pairwise generic by including into the quantum formula an encoding of Equation (4). This establishes the equivalence to strong satisfiability “ $\exists \vec{Y} : g(\vec{Z}, \vec{U}) = 1$ ”. Reduction to the case of weak satisfiability proceeds as in Example 24c). \square

3 Computational Complexity of Satisfiability

An algebraic structure gives rise to various decision problems: Are two given terms equivalent in the sense that they coincide for all assignments? Or is there an assignment for which they differ? Or is there one for which they agree? Algorithmically solving these problems are the basis of automatic simplifications necessary for computer algebra systems. As an illustration, consider the following (meta-)

Example 26. Fix a finite (say, two-element) set of variables and the family of terms they generate over the structure of non-commutative groups, i.e. with the operations “ \circ ” and its inverse “ $^{-1}$ ”. Fix furthermore a finite set of equational relations (‘rules’) among some terms. The word problem is the question whether two given terms f, g are, subject to the rules, equivalent.

Note that if they are, trial-and-error over repeated application of appropriate rules will eventually reveal them equivalent. The difficult part of the task is to correctly detect with certainty and in finite time the case $f \neq g$. And by the famous work of NOVIKOV (1955) and, independently, BOONE (1958), this question provably is in general not decidable.

The same problem for the more general class of monoids had already been shown undecidable by POST in 1947; see e.g. [ABR92]. For a decidable problem, it makes sense to investigate their algorithmic complexity, cmp. e.g. [Klim09].

In the synthetic context of quantum logic, decidability is known for the word problem for free ortholattices [Brun76, Mein05] and undecidable for free modular lattices [Herr83] but remains an open challenge in the orthomodular [Herr87] as well as in the modular-ortho case (cf. [HMR05]). For free orthomodular lattices over two generators, the problem is trivially decidable; in fact a nice online implementation can be found at [Hyck04] and some considerations for modest generalization in [HyNa05]. In the geometric context of (modular) projection lattices of finite von Neumann algebra factors, decidability has been shown in [Herr10]. For the special case of $\text{Gr}(\mathcal{H})$ with finite-dimensional \mathcal{H} , the word problem (i.e. tautology of a formula, equivalence to 1, weak unsatisfiability of its complement) has been pointed out as decidable in [DHMW05, SECTION 3]; whereas the case of $\text{Gr}(\mathcal{H})$ for an infinite-dimensional Hilbert space \mathcal{H} remains open. (The dissertation [Denn78] for instance considers only words *without* negation.)

The present and following sections investigate more closely the computational complexity of quantum logic satisfiability: in the standard Turing machine model, that is in terms of famous complexity classes \mathcal{P} , \mathcal{NP} , and PSPACE ; cf. e.g. [Papa94]. For instance from a standard text-book analysis of Gaussian Elimination, we record the following

Observation 27. For $0 \leq k, \ell \leq d$ and given matrices $A \in (\mathbb{Q} + i\mathbb{Q})^{k \times d}$ and $B \in (\mathbb{Q} + i\mathbb{Q})^{\ell \times d}$ of maximum rank (namely k and ℓ , respectively) a Turing machine can, in time polynomial in the bitlength of A and B , calculate matrices C_1, C_2, C_3 of full rank with

- 1) $\text{range}(C_1) = \neg \text{range}(A)$
- 2) $\text{range}(C_2) = \text{range}(A) \vee \text{range}(B)$
- 3) $\text{range}(C_3) = \text{range}(A) \wedge \text{range}(B)$.

In particular, given matrices $A_1, \dots, A_n, Z \in (\mathbb{Q} + i\mathbb{Q})^{d \times d}$ and a formula $f(X_1, \dots, X_n)$, a Turing machine can in polynomial time decide whether $f(\text{range}(A_1), \dots, \text{range}(A_n)) \geq \text{range}(Z)$ holds.

Some arguments and intermediate results will occasionally also refer to (variants of) the Blum-Shub-Smale model of computation. Such a BSS machine (over \mathbb{R} without order) may be regarded as a Turing machine with the capability to store and operate not on bits and/or integers but on real numbers ($+, -, \times, \div, \leq$) exactly and in unit time [BSS89, BCSS98]. This turns out to considerably simplify many arguments and opening to a rich variety of technical tools, e.g. in Fact 59b) and the proof of Theorem 62a+b). We consider BSS machines both

with and without finitely-many pre-stored real constants like, e.g., $\sqrt{2}$ or π . Another review of Gaussian Elimination reveals:

Observation 28. *Given a formula $f(X_1, \dots, X_n)$ and (real and imaginary parts of the coefficients of) matrices $A_1, \dots, A_n, Z \in \mathbb{C}^{d \times d}$, a BSS machine devoid of constants can, in time polynomial in d and the length of f , decide whether $f(\text{range}(A_1), \dots, \text{range}(A_n)) \geq \text{range}(Z)$ holds.*

Of course, the d^2 entries of a matrix $A \in \mathbb{F}^{d \times d}$ represent the subspace $\text{range}(A) \in \text{Gr}(\mathbb{F}^d)$ only with certain redundancy. As a least-redundant encoding of a k -dimensional subspace, we shall in Section 4.7 below employ the $d \cdot (d - k)$ -tuple of its Plücker Coordinates.

3.1 Satisfiability of Conjunctive Formulas is Polynomial-Time Decidable

Over Booleans, satisfiability of a given formula f is well-known \mathcal{NP} -complete, that is

- i) it can be decided in time polynomial in the length $|f|$ of f by a *nondeterministic* Turing machine (belongs to \mathcal{NP});
- ii) it is hardest among the problems decidable by a nondeterministic polynomial-time Turing machine

in the sense that any other problem in \mathcal{NP} can be reduced to it in (deterministic) polynomial time. In fact the famous Cook-Levin Theorem states that satisfiability still remains \mathcal{NP} -complete when restricting to formula f in *conjunctive* form

$$f(X_1, \dots, X_n) = \bigwedge_i \bigvee_j Y_{i,j}, \quad Y_{i,j} \in \{X_1, \dots, X_n, \neg X_1, \dots, \neg X_n\} . \quad (8)$$

Surprisingly, quantum satisfiability of quantum formulas in conjunctive form can even be decided in deterministic polynomial time, i.e. is considerably *easier* than in the Boolean case:

Theorem 29. *Consider a formula $f(X_1, \dots, X_n)$ in conjunctive form $f = \bigwedge_i K_i$, where $K_i = \bigvee_{j=1}^{J_i} Y_{i,j}$ are called **clauses** and the **literals** $Y_{i,j} \in \{X_1, \neg X_1, \dots, X_n, \neg X_n\}$ satisfy $Y_{i,j} \notin \{Y_{i,k}, \neg Y_{i,k}\}$ for $j \neq k$ (i.e. within one clause refer to distinct variables). Given such f and the parity of $d > 1$, a Turing machine can decide in polynomial time whether f is [weakly] satisfiable in $\text{Gr}(\mathbb{F}^d)$, independent of \mathbb{F} .*

Admittedly, this is not so surprising any more when noticing that normal form means an actual semantic restriction to the expressiveness of quantum formulas in Item d) of the following

Lemma 30. *Let f be in conjunctive form as in Theorem 29.*

- a) *If $J_i \geq 2$ for each i (i.e. if all clauses contain at least 2 literals), then f is satisfiable in even dimensions d ; and weakly satisfiable for d odd.*
- b) *If $J_i \leq 2$ for each i , then f is satisfiable over Booleans iff it is satisfiable in any odd dimension d .*
- c) *If $J_i \geq 3$ for each i , then f is satisfiable in all dimensions $d > 1$.*
- d) *$(X \wedge Y) \vee (X \wedge \neg Y)$ is a formula in disjunctive form not equivalent over $\text{Gr}(\mathbb{F}^2)$ to any formula in conjunctive form.*

- Proof.* a) Let $(V_1, \dots, V_n) \in \text{Gr}_d(\mathbb{F}^{2d})^n$ be pairwise generic (Lemma 10). Then $f(V_1, \dots, V_n) = 1$. Indeed already each two-literal clause $X_i \vee X_j$ evaluates to $1 = \mathbb{F}^{2d}$ as well as $X_i \vee \neg X_j$ and $\neg X_i \vee \neg X_j$.
- Embedding V_i from $\text{Gr}_d(\mathbb{F}^{2d})$ to $\text{Gr}_d(\mathbb{F}^{2d+1}) \ni \tilde{V}_i := V_i \times \{0\}$ yields $\tilde{V}_i \vee \tilde{V}_j, \tilde{V}_i \vee \neg \tilde{V}_j, \neg \tilde{V}_i \vee \neg \tilde{V}_j \supseteq \mathbb{F}^{2d} \times \{0\}$; hence $f(\tilde{V}_1, \dots, \tilde{V}_n) \neq 0$. (Compare also Section 4.1 on dimensional heredity of weak satisfiability.)
- b) If f is satisfiable over $\{0, 1\} = \text{Gr}(\mathbb{F}^1)$, then so is it over $\{\{0\}, \mathbb{F}^d\} \subseteq \text{Gr}(\mathbb{F}^d)$. For the converse, recall that Boolean satisfiability of 2SAT is reducible to the **implication graph**: For a given formula $f(X_1, \dots, X_n)$, G_f has vertices $\{X_1, \dots, X_n, \neg X_1, \dots, \neg X_n\}$ and directed edges (Y_i, Y_j) and $(\neg Y_j, \neg Y_i)$ between literals Y_i, Y_j whenever $\neg Y_i \vee Y_j$ is among the clauses of f —because the choice $Y_i = 1$ would require $Y_j = 1$ in order to satisfy this clause, and $Y_j = 0$ would require $Y_i = 0$ by duality. Hence, by construction, f is satisfiable over $\{0, 1\}$ iff each vertex (literal) Y_i in G_f is not strongly connected to its complement $\neg Y_i$.
- This classical observation carries over to odd dimensions as follows: In a clause $\neg Y_i \vee Y_j$, a satisfying assignment with $\dim(Y_i) > d/2$ requires $\dim(Y_j) > d/2$ (Example 7g); and dually $\dim(Y_j) < d/2$ implies $\dim(Y_i) < d/2$. Hence, the strong disconnectedness condition of G_f is also necessary for the satisfiability of f in any odd dimension.
- c) One can partially generalize Lemma 10 to odd dimensions d (and Lemma 34 below shall indeed formally establish) that there exist subspaces $V_1, \dots, V_n \in \text{Gr}_{(d-1)/2}(\mathbb{F}^d)$ such that, for any $i < j < k$, it holds $1 = V_i \vee V_j \vee V_k = V_i \vee \neg V_j = \neg V_i \vee \neg V_j = \neg V_i \vee \neg V_j$. In particular, any three-literal clause evaluates to 1; hence $f(V_1, \dots, V_n) = 1$.
- d) Since a possible conjunctive form g of $f = (X \wedge Y) \vee (X \wedge \neg Y)$ must coincide with f when restricting to the Booleans $\{0\}, \mathbb{F}^2 \in \text{Gr}(\mathbb{F}^2)$, g has to be either X or $(X \vee Y) \wedge (X \vee \neg Y)$. But neither of them is equivalent to f over $\text{Gr}(\mathbb{F}^2)$. \square

Proof (Theorem 29). First consider the one-element clauses K_i , i.e. those with $J_i = 1$: If $K_i = Y_{i,j} = \neg K_{i'}$ for some i, i' , then obviously $f = 0$ is not even weakly satisfiable over any $\text{Gr}(\mathcal{H})$. This case can be computationally detected in polynomial time.

Otherwise, one may safely eliminate these one-element clauses by assigning (and now indeed consistently) $K_i := 1$ for all i with $J_i = 1$, and apply to the remaining clauses the trivial simplifications “ $X \vee 0 = X$ ” and “ $X \vee 1 = 1$ ”, in the latter case possibly removing clauses that have become tautology. Then, again, identify clauses which now may have been reduced to one literal and repeat the above process: an at most polynomial number of times. In the end we have arrived at a simplified, equivalent formula containing only clauses with at least two literals: $J_i \geq 2$.

By Lemma 30a), this is satisfiable over any even dimension; and weakly satisfiable over any dimension $d > 1$. In the odd-dimensional case of strong satisfiability, consider the subformula g consisting of those clauses with $J_i = 2$. Use the classical algorithm for 2SAT to decide in polynomial time whether g admits a satisfying Boolean assignment $X_1, \dots, X_n \in \{0, 1\}$. If not, according to Lemma 30b), g (and hence also f) is not satisfiable over \mathbb{F}^{2d-1} either.

Otherwise consider $V_1, \dots, V_n \in \text{Gr}_{(d-1)/2}(\mathbb{F}^d)$ as in the proof of Lemma 30c). Recall the proof of Lemma 30b) to observe that setting $\tilde{X}_i := V_i$ for $X_i = 0$ and $\tilde{X}_i := \neg V_i$ for $X_i = 1$ yields another satisfying assignment of g , that is the 2-element clauses of f —in fact an assignment which, similar to the proof of Lemma 30c), satisfies the remaining clauses of f as well. \square

3.2 2D Satisfiability is Nondeterministically Polynomial-Time Complete

Without the restriction to conjunctive form, the classical complexity of Boolean satisfiability holds also for 2D quantum satisfiability.

Theorem 31. *a) 2D [weak] quantum logic satisfiability belongs to the complexity class \mathcal{NP} , i.e. can be decided in polynomial time by a nondeterministic Turing machine, independent of \mathbb{F} .
b) Again independent of \mathbb{F} , 2D satisfiability of a formula is \mathcal{NP} -hard:
Given a Boolean formula f in conjunctive form, a Turing machine can in polynomial time calculate another formula g having the property that g is [weakly] satisfiable over \mathbb{F}^2 iff f is over $\{0, 1\}$.*

Proof (Theorem 31).

- a) By (the proof of) Lemma 10d), a Turing machine can in time polynomial in n prepare (and in particular use only polynomially many bits in order to describe) a pairwise generic n -tuple (V_1, \dots, V_n) . Then nondeterministically guess $W_i \in \{0, 1, V_1, \neg V_1, \dots, V_n, \neg V_n\}$ for each $i = 1, \dots, n$. Finally evaluate, by Observation 27 again in polynomial time, $f(W_1, \dots, W_n)$ and check whether the result coincides with 1 [is different from 0]. According to Corollary 12c), such \vec{W} exists iff f is [weakly] satisfiable.
- b) Let $g(X_1, \dots, X_n) := f(X_1, \dots, X_n) \wedge \bigwedge_{i < j} C(X_i, X_j)$. If $(x_1, \dots, x_n) \in \{0, 1\}^n$ is a satisfying assignment of f , then it is also one of g because of $C(x_i, x_j) = 1$. Now suppose conversely that $g(\vec{W}) \neq 0$ for $W_1, \dots, W_n \in \text{Gr}(\mathbb{F}^2)$. Then $C(W_i, W_j) \neq 0$ requires (Example 13a) all $W_i \in \{0, 1, W_j, \neg W_j\}$; that is $\deg(\vec{W}) \leq 1$. Hence Example 13b) implies that g also has a Boolean satisfying assignment; which is one of f , too. \square

4 Higher but Fixed Dimensions

Most of the above results were concerned with 2D quantum logic. Recall (Lemma 10e) that the notion of pairwise genericity (Definition 9) makes sense only in even dimensions. Theorem 35c) below will generalize Proposition 19b) to higher dimensions. This is not as straightforward as it may first seem:

Example 32. Let $X := \text{lspan} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $Y := \text{lspan} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $Z := \text{lspan} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then the following quantum logic formula with three constants does not satisfy Equation (7):

$$f(\cdot, X, Y, Z) : \text{Gr}(\mathbb{F}^3) \ni W \mapsto \neg(C(X, W) \wedge C(Y, W) \wedge C(Z, W) \wedge \neg W)$$

Specifically, $f(W) = Z \vee W \neq 1$ for $W := \text{lspan} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \neq 0$.

Indeed, $W \perp X$ implies $C(X, W) = 1$; similarly $C(Y, W) = 1$. And $C(Z, W)$ evaluates to $\neg Z \wedge \neg W \neq 0$. \square

Let us first adapt the notion of genericity (Definition 9):

Definition 33. Consider $U_1, \dots, U_d \in \text{Gr}_1(\mathbb{F}^d)$. Call (U_1, \dots, U_d) generic if it holds

$$1 = \bigvee_{i=1}^d U_i \quad \text{and} \quad 0 = U_i \cap \neg U_j \quad \forall i, j. \quad (9)$$

An set $\mathcal{U} \subseteq \text{Gr}_1(\mathbb{F}^d)$ is generic if $\text{Card}(\mathcal{U}) \geq d$ and each d -element subset of \mathcal{U} is generic in the above sense.

Observe that, since $\dim(U_i) = 1$, $U_i \cap \neg U_k = 0$ is equivalent to the lines not being perpendicular: $U_i \not\perp U_k$. Hence our notion is justified by [HaSv96]. Also, in dimension $d = 2$, genericity coincides with pairwise genericity in the sense of Definition 9. Generalizing Example 7d), we have

Lemma 34. *a) Let $0 \leq t_1 < t_2 < \dots < t_n \in \mathbb{R}$. Then the following family of subspaces is generic:*

$$U_i := \text{lspan } \vec{u}_i, \quad \vec{u}_i := (1, t_i, t_i^2, \dots, t_i^{d-1}) .$$

- b) *Let (U_1, \dots, U_d) be generic. Then it holds $\bigwedge_{i=1}^{d-1} C(U_i, U_d) = 0$.*
c) *Let $\mathcal{U} \subseteq \text{Gr}_1(\mathbb{F}^d)$ be generic of $\text{Card}(\mathcal{U}) \geq 2d - 1$ and $V \in \text{Gr}_1(\mathbb{F}^d)$. Then there exist $U_1, \dots, U_{d-1} \in \mathcal{U}$ such that (U_1, \dots, U_{d-1}, V) is generic.*
d) *Let $\mathcal{U} \subseteq \text{Gr}_1(\mathbb{F}^d)$ be generic of $\text{Card}(\mathcal{U}) \geq 2d - 1$ and $0 \neq V \in \text{Gr}(\mathbb{F}^d)$. Consider $\Pi(X, Z) := Z \wedge (X \vee \neg Z)$, the projection of X to Z . Then there exist $U_1, \dots, U_d \in \mathcal{U}$ such that $\Pi(V, U_i) = U_i$ for $i = 1, \dots, d$.*
e) *Let $\text{generic}_d(Y_1, \dots, Y_d) := \bigvee_{i=1}^d Y_i \wedge \bigwedge_{1 \leq i, j \leq d} (\neg Y_i \vee Y_j) \wedge \bigwedge_{j=1}^d (\neg Y_j \vee \neg \bigvee_{i \neq j} Y_i)$;
 $\text{generic}_d(Y_1, \dots, Y_n) := \bigwedge_{1 \leq i_1 < \dots < i_d \leq n} \text{generic}_d(Y_{i_1}, \dots, Y_{i_d})$. Then $\text{generic}_d(\vec{Y}) = 1 \Leftrightarrow \vec{Y}$ is generic.*

Proof. a) Consider any d vectors \vec{u}_i as above and collect them as columns to an the $d \times d$ -matrix A . This is of Vandermonde form with determinant $\prod_{i < j}^d (t_i - t_j) \neq 0$, hence regular with $\mathbb{F}^d = \text{range}(A) = U_1 \vee \dots \vee U_d$. Moreover $\langle \vec{u}_i, \vec{u}_j \rangle \geq 1$ shows $U_i \not\perp U_j$.

b) Consider $C(U_i, U_j) = (U_i \wedge U_j) \vee (U_i \wedge \neg U_j) \vee (\neg U_i \wedge U_j) \vee (\neg U_i \wedge \neg U_j)$. The middle two terms are $= 0$ by Definition; and so is the first one for $i \neq j$, because $1 = \dim(U_i) = \dim(U_j)$ and $d = \dim(\bigvee_k U_k)$. Therefore $\bigwedge_{i=1}^{d-1} C(U_i, U_d) = \bigwedge_{i=1}^{d-1} \neg U_i \wedge \neg U_d = 0$ by the complement of Equation (9).

c) Observe that there are at most $d - 1$ elements $U \in \mathcal{U}$ such that $V \subseteq \neg U$: $0 \neq V \subseteq \neg U_1 \wedge \neg U_2 \wedge \dots \wedge \neg U_d$ would imply $1 \neq U_1 \vee U_2 \vee \dots \vee U_d$ contradicting that (U_1, \dots, U_d) is generic. So consider the subset $\mathcal{U}' := \{U \in \mathcal{U} : 0 = V \wedge \neg U\}$ of $\text{Card}(\mathcal{U}') \geq d$; w.l.o.g. $= d$. Since $\mathcal{U}' = \{U_1, \dots, U_d\}$ is generic by hypothesis, $\bigvee \mathcal{U}' = 1$ holds; hence Steinitz' Lemma yields some j such that $V \vee U_1 \vee \dots \vee U_{j-1} \vee U_{j+1} \vee \dots \vee U_d = 1$. Now observe that $(V, U_1, \dots, U_{j-1}, U_{j+1}, \dots, U_d)$ is generic.

d) Since any d distinct $U_1, \dots, U_d \in \mathcal{U}$ already span entire \mathbb{F}^d , V cannot be perpendicular to all U_1, \dots, U_d ; i.e. there exists $1 \leq i \leq d$ with $\Pi(V, U_i) \neq 0$. Repeating the same argument on $U_1, \dots, U_{i-1}, U_{i+1}, \dots, U_d, U_{d+1}$, we iteratively find $i_1 < \dots < i_d \leq 2d - 1$ with $0 \neq \Pi(V, U_{i_j}) \subseteq U_{i_j}$, i.e. $\Pi(V, U_{i_j}) = U_{i_j}$.

e) Generic (Y_1, \dots, Y_d) satisfies the first two terms by definition; and all 1D Y_1, \dots, Y_d span the whole space but any $d - 1$ cannot, hence Y_j must be disjoint from $\bigvee_{i \neq j} Y_i$. Conversely, $\neg Y_i \vee Y_j = 1$ implies $\dim(Y_i) \leq \dim(Y_j)$, hence all Y_i have the same dimension; which must be 1 by $\bigvee_i Y_i = 1$ and $Y_j \wedge \bigvee_{i \neq j} Y_i = 0$. \square

Theorem 35. *a) Let $U_1, \dots, U_d \in \text{Gr}_1(\mathbb{F}^d)$ be generic. Then $C(X, U_i) = 1$ for all $i = 1, \dots, d$ implies $X \in \{0, 1\}$.*

b) *Consider the formula $\text{generic}_d(\vec{Y}) \wedge \bigwedge_{1 \leq i \leq d} C(X, Y_i)$. Then generic \vec{Y} , together with both $X = 0$ and $X = 1$, constitute a satisfying assignment; and conversely any satisfying assignment $X, \vec{Y} \in \text{Gr}(\mathbb{F}^d)$ has $X \in \{0, 1\}$ (and \vec{Y} generic).*

c) Let $U_1, \dots, U_{2d-1} \in \text{Gr}_1(\mathbb{F}^d)$ be generic. Then the formula with constants

$$f := \tilde{f}(\cdot, U_1, \dots, U_{2d-1}) : \text{Gr}(\mathbb{F}^d) \ni X \mapsto \bigvee_{i=1}^{2d-1} \Pi(X, U_i)$$

attains only values $\{0, 1\}$ and satisfies Equation (7).

- d) In any even dimension $d \geq 4$, there exist $U_1, U_2, U_3 \in \text{Gr}_{d/2}(\mathbb{F}^d)$ such that $C(X, U_1) \wedge C(X, U_2) \wedge C(X, U_3)$ is satisfied precisely for $X \in \{0, 1\}$.
- e) For each $d \in \mathbb{N}$ and independent of \mathbb{F} , there exist formulas $f(X, \vec{Y})$ and $g(X, \vec{Z})$ such that, over $\text{Gr}(\mathbb{F}^d)$, it holds:

$$X = 1 \quad \Leftrightarrow \quad \exists \vec{Y} : f(X, \vec{Y}) \neq 0 \quad \Leftrightarrow \quad \forall \vec{Z} : g(X, \vec{Z}) \neq 0 .$$

Item b) means that Boolean are definable over $\text{Gr}(\mathbb{F}^d)$ by an existentially quantified formula for each fixed d ; and also by a quantifier-free formula with d constants; whereas Item d) amounts to definability with three constants. Item e) generalizes Example 24c); cmp. also [HMR05, LEMMA 5].

Proof. a) $1D \ U_i = (U_i \wedge X) \vee (U_i \wedge \neg X)$ requires $U_i \subseteq X$ or $U_i \subseteq \neg X$. That is, there exists $I \subseteq [d] := \{1, \dots, d\}$ such that $U_i \subseteq X$ for $i \in I$ and $U_j \subseteq \neg X$ for $j \notin I$. Consequently, $U_I := \bigvee_{i \in I} U_i \subseteq X$ and $U_{[d] \setminus I} = \bigvee_{j \notin I} U_j \subseteq \neg X$; whereupon $U_I \perp U_{[d] \setminus I}$ follows. But $U_i \not\subseteq U_j$ requires $I = [d]$ or $I = \emptyset$, i.e. $X = 1$ or $\neg X = 1$.

b) For a satisfying assignment (X, \vec{Y}) , \vec{Y} is generic according to Lemma 34e). So the last term requires $X \in \{0, 1\}$ by a). The converse is straight-forward.

c) Since $\Pi(0, Z) = 0$, $f(0) = 0$ follows. Whereas for $X \neq 0$, by Lemma 34d), there exist $i_1 < \dots < i_d$ with $\Pi(U_{i_j}, X) = U_{i_j}$; hence $f(X) \geq \bigvee_{j=1}^d U_{i_j} = 1$ according to Definition 33.

d) is deferred to Theorem 86d).

e) Let $\neg g(\neg X, \vec{Z}) := \text{generic}_d(U_1, \dots, U_{2d-1}) \wedge \bigvee_{i=1}^{2d-1} \Pi(X, U_i)$. According to c), there exist (necessarily generic) \vec{U} such that this formula evaluates to 1 iff $X \neq 0$; and thus $g(X, \vec{Y})$ to 0 iff $X \neq 1$: the contraposition.

Our definition of $f(X, \vec{Y})$ anticipates Fact 37b+d) below, namely as the restriction $\psi_d(\vec{Y})|_X$. Then for $\dim(X) = d$, there exist \vec{Y} with $0 < \dim f(X, \vec{Y})$; whereas for $\dim(X) < d$ it holds $0 = \dim f(X, \vec{Y})$. \square

4.1 On the Dimensional Heredity of Satisfiability

As already mentioned, [DHMW05, Hagg07] has succeeded in constructing formula ψ_d weakly satisfiable in $\text{Gr}(\mathbb{F}^{d+1})$ but not in $\text{Gr}(\mathbb{F}^d)$. Conversely it seems intuitively clear that any formula $f(X_1, \dots, X_n)$ admitting a weakly satisfying assignment (U_1, \dots, U_n) in dimension d also does so in dimension $k > d$; compare the proof of [DHMW05, LEMMA 5]. We strengthen it to the following

Lemma 36. *Let \mathcal{H} be a Hilbert space and $X_1, \dots, X_n, Z \in \text{Gr}(\mathcal{H})$. Suppose that Z commutes with each X_i . Then it holds $\Xi_Z(f; X_1 \wedge Z, \dots, X_n \wedge Z) = \Xi_{\mathcal{H}}(f; X_1, \dots, X_n) \wedge Z$.*

In particular if $X_1, \dots, X_n \in \text{Gr}(Z)$ is a weakly satisfying assignment of f in Z , then it is also one of f in \mathcal{H} . Note also that, for formulas with constants $f(\vec{c}, \vec{X})$, Lemma 36 yields $\Xi_{\mathcal{H}}(f; \vec{c}, \vec{X}) \wedge Z = \Xi_Z(f; \vec{c} \wedge Z, \vec{X} \wedge Z)$ which is generally different from $\Xi_Z(f; \vec{c}, \vec{X} \wedge Z)$ even in case $c_i \subseteq Z$.

Proof (Lemma 36). Recall that commutativity implies distributivity (Fact 4); hence, for example, $(X_i \vee X_j) \wedge Z = (X_i \wedge Z) \vee (X_j \wedge Z)$ and, by de Morgan, $\neg X_i \wedge Z = \neg(X_i \wedge Z) \wedge Z$ hold. Moreover commutativity w.r.t. Z extends from \vec{X} to all quantum logic expressions over \vec{X} (Fact 4c); and in particular to all subexpressions of f .

Put differently, Definition 5b) straight-forwardly extends to the ortholattice $[0, Z] \cup [\neg Z, 1]$ as direct product of the intervals $[0, Z]$ and $[\neg Z, 1]$. \square

(Strong) satisfiability on the other hand turns out to fail dimensional heredity in general. But first we provide some additional tools, based on the following important

Fact 37. *For an n -variate formula f , let*

$$\text{maxdim}_{\mathbb{F}}(f, d) := \max \left\{ \dim f(\vec{X}) : X_1, \dots, X_n \in \text{Gr}(\mathbb{F}^d) \right\} .$$

- a) 3-variate formula $h(p, q, r) := (p \vee q) \wedge (p \vee r) \wedge (\neg q \vee \neg r) \wedge \neg p$ has $\text{maxdim}_{\mathbb{F}}(h, d) = \lfloor d/2 \rfloor$.
- b) For formulas $f(\vec{X})$ and $g(\vec{Y})$ let the *restriction* $f(\vec{X})|_{g(\vec{Y})}$ be defined by replacing in f each X_i with $X_i \wedge g(\vec{Y})$ and each $\neg X_i$ with $\neg(X_i \wedge g(\vec{Y})) \wedge g(\vec{Y})$, where w.l.o.g. $f(\vec{X}, \neg \vec{X})$ is presumed free of negations (de Morgan). Then $\text{maxdim}_{\mathbb{F}}(f|_g, d) = \text{maxdim}_{\mathbb{F}}(f, \text{maxdim}_{\mathbb{F}}(g, d))$.
- c) Let $\Pi(X, Z) := Z \wedge (X \vee \neg Z)$ denote the projection of X to Z . Then it holds $\dim(\Pi(X, Z)) \leq \dim(Z)$. In particular, $\psi_{1,2}(X, Z) := \Pi(\Pi(\Pi(X, Z), X), \neg Z)$ has $\dim(\psi_{1,2}(X, Z)) \leq \min \{ \dim(Z), \dim(\neg Z) \}$; and there exist $X, Z \in \text{Gr}(\mathbb{F}^d)$ with $\dim(\psi_{1,2}(X, Z)) = \lfloor d/2 \rfloor$.
- d) To each $d \in \mathbb{N}$ there exists a formula ψ_d independent of \mathbb{F} in $\mathcal{O}(d)$ variables and of length $2^{\mathcal{O}(d)}$ such that $\text{maxdim}_{\mathbb{F}}(\psi_d, d-1) = 0$ but $\text{maxdim}_{\mathbb{F}}(\psi_d, d) > 0$.

Claim a) is due to [DHMW05], b+d) due to [Hagg07]; c) taken from [Hagg09, SLIDE #13]. Concerning the length ℓ_d of ψ_d , observe that $\psi_{2d}(\vec{X}, \vec{Y})$ is constructed as $\psi_d(\vec{X})|_{\psi_d(\vec{Y})}$, i.e. each of the $\mathcal{O}(\ell_d)$ occurrences of X_i or $\neg X_i$ in ψ_d is replaced by a formula of length $\mathcal{O}(\ell_d)$ —leading to $\ell_{2d} = \mathcal{O}(\ell_d^2)$. We shall improve that in Theorem 68b) and present in Lemma 40f) a simpler proof of [Hagg07, THEOREM 1].

As noted in [Hagg07, between Definitions 2 and 3] for the case $\mathbb{F} = \mathbb{C}$, $\text{maxdim}(f, Z) := \max \{ \dim(f(\vec{X}) : X_1, \dots, X_n \in \text{Gr}(Z)) \}$ depends only on $\dim(Z)$; this follows formally, and also for a larger class of fields, from the below Lemma 40c).

Lemma 38. *Call linear map $U : \mathcal{H} \rightarrow \mathcal{H}$ scaled-unitary if there exists $\lambda > 0$ such that $U^\dagger \cdot U = \lambda \text{id}$ where U^\dagger denotes the adjoint of U (i.e. $(u_{ij})^\dagger = (u_{ji}^*)$ in the finite-dimensional case.) We write $\tilde{\mathcal{U}}(\mathcal{H})$ for the set of scaled-unitary matrices and $U \cdot X := \{U \cdot \vec{x} : \vec{x} \in X\} \in \text{Gr}(\mathcal{H})$ for $X \in \text{Gr}(\mathcal{H})$.*

- a) Linear $U : \mathcal{H} \rightarrow \mathcal{H}$ is unitary-like iff it holds

$$\forall \vec{x}, \vec{y} \in \mathcal{H} : \quad \langle \vec{x}, \vec{y} \rangle = 0 \Leftrightarrow \langle U \cdot \vec{x}, U \cdot \vec{y} \rangle = 0 . \quad (10)$$

- b) If $U \in \tilde{\mathcal{U}}(\mathcal{H})$, then also $U^{-1}, U^\dagger \in \tilde{\mathcal{U}}(\mathcal{H})$ and $\|U \cdot \vec{x}\|^2 = \lambda \|\vec{x}\|^2$ for the Euclidean norm squared $\|\vec{x}\|^2 = \sum_i |x_i|^2$.
- c) Let h denote a quantum logic formula in n variables and $X_1, \dots, X_n \in \text{Gr}(\mathcal{H})$. Then for $U \in \tilde{\mathcal{U}}(\mathcal{H})$ it holds $h(U \cdot X_1, \dots, U \cdot X_n) = U \cdot h(X_1, \dots, X_n)$. In other words: $\text{Gr}(\mathbb{F}^d) \ni X \mapsto U \cdot X \in \text{Gr}(\mathbb{F}^d)$ constitutes an ortho-isomorphism.

- d) Let $\vec{v}_1, \dots, \vec{v}_d \in \mathbb{F}^d$ be pairwise orthogonal and similarly for $\vec{w}_1, \dots, \vec{w}_d \in \mathbb{F}^d$ with $\|\vec{v}_i\|^2 = \lambda \|\vec{w}_i\|^2$ for $i = 1, \dots, n$ and one $\lambda > 0$. Then there exists $U \in \tilde{\mathcal{U}}(\mathcal{H})$ with $U : \vec{v}_i \mapsto \vec{w}_i$ for all $i = 1, \dots, d$.
- e) Let $x, y, u, v \in \mathbb{Z}$. If $x^2 - y^2$ is even, then it is even a multiple of 4. If $2u^2 + x^2 = 2v^2 + y^2$, then necessarily $x, y, u, v = 0$.
- f) Let $X := \mathbb{F}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \text{Gr}_1(\mathbb{Q}^3)$ and $Y := \mathbb{F}\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right) \in \text{Gr}_1(\mathbb{Q}^3)$. There exists no $U \in \tilde{\mathcal{U}}(\mathbb{Q}^3)$ with $U \cdot X = Y$. More precisely, $\neg Y = \{(u, u, x) : u, x \in \mathbb{Q}\} \in \text{Gr}_2(\mathbb{Q}^3)$ has no basis of vectors of common length.

Proof. a) For unitary-like U , Equation (10) is readily verified. Conversely suppose there exists nonzero $\vec{x} \in \mathcal{H}$ with $\vec{z} := U^\dagger \cdot U \cdot \vec{x}$ not a multiple of \vec{x} . Then \vec{x}, \vec{z} together span a plane in \mathcal{H} containing some nonzero \vec{y} orthogonal to \vec{x} but not to \vec{z} :

$$0 \neq \langle \vec{z}, \vec{y} \rangle = \langle U^\dagger \cdot U \cdot \vec{x}, \vec{y} \rangle = \langle U \cdot \vec{x}, U \cdot \vec{y} \rangle :$$

contradicting Equation (10). Hence $U^\dagger \cdot U = \lambda \text{id}$ for some $\lambda \in \mathbb{C}$ is necessary. In fact

$$\lambda \langle \vec{x}, \vec{x} \rangle = \langle U^\dagger \cdot U \cdot \vec{x}, \vec{x} \rangle = \|U \cdot \vec{x}\|^2 \geq 0 \quad (11)$$

reveals $\mathbb{R} \ni \lambda \geq 0$. Moreover $U \cdot \vec{x} = 0$ for some $\vec{x} =: \vec{y} \neq 0$ would contradict Equation (10): hence $\lambda \neq 0$.

- b) Observe $U^\dagger = \lambda \cdot U^{-1}$. Scaled isometry has already been shown in Equation (11).
- c) By a), $U \cdot \vec{y} \in U \cdot (\neg X) \Leftrightarrow \langle \vec{y}, \vec{x} \rangle = 0 \ \forall \vec{x} \in X \Leftrightarrow \langle U \cdot \vec{y}, U \cdot \vec{x} \rangle = 0 \ \forall \vec{x} \in X \Leftrightarrow \langle U \cdot \vec{y}, \vec{v} \rangle = 0 \ \forall \vec{v} \in U \cdot X \Leftrightarrow U \cdot \vec{y} \in \neg(U \cdot X)$. Hence $U \cdot (\neg X) = \neg(U \cdot X)$; $U \cdot (X \vee Y) = (U \cdot X) \vee (U \cdot Y)$ and $U \cdot (X \wedge Y) = (U \cdot X) \wedge (U \cdot Y)$ follow from mere linearity, bijectivity, and continuity.

Now apply structural induction on the length of the formula under consideration.

- d) $U : \vec{v}_i \mapsto \vec{w}_i$ welldefines a linear map $U : \mathbb{F}^d \rightarrow \mathbb{F}^d$. Moreover $\vec{x} = \sum_i \lambda_i \vec{v}_i$ and $\vec{y} = \sum_j \mu_j \vec{v}_j$ have

$$\langle \vec{x}, \vec{y} \rangle = \sum_{i,j} \lambda_i \mu_j^* \langle \vec{v}_i, \vec{v}_j \rangle = \sum_i \lambda_i \mu_i^* \|\vec{v}_i\|_2 \stackrel{(*)}{=} \sum_i \lambda_i \mu_i^* \lambda \|\vec{w}_i\|_2 = \lambda \langle U \cdot \vec{x}, U \cdot \vec{y} \rangle$$

with $(*)$ by hypothesis. Hence U is scaled-unitary according to a).

- e) Since prime 2 divides $x^2 - y^2 = (x + y)(x - y)$, it must divide $x + y$ or $x - y$. In either case x, y must have the same parity; hence 2 in fact divides both $x + y$ and $x - y$. Since 2 divides $2(v^2 - u^2) = x^2 - y^2$, we have just seen that also 4 divides $x^2 - y^2 = 2(v^2 - u^2)$, i.e. 2 divides $v^2 - u^2$; thus 4 divides $v^2 - u^2$, and 8 divides $2(v^2 - u^2) = x^2 - y^2$. Hence every power of 2 divides $x^2 - y^2$: this is possible only for $x^2 - y^2 = 0$.
- f) Suppose the contrary. Then, according to a+b), U maps the normed pairwise-orthogonal vectors $(1, 0, 0) \in X$, $(0, 1, 0), (0, 0, 1) \in \neg X$ to pairwise-orthogonal vectors $\vec{a} \in Y$, $\vec{b}, \vec{c} \in \neg Y$ of common norm $\lambda = \|\vec{a}\|^2 = \|\vec{b}\|^2 = \|\vec{c}\|^2$. It thus suffices to prove the second claim. So let $\vec{a} = (u, u, x)$ and $\vec{b} = (v, v, y) \in \mathbb{Q}^3$ with $2u^2 + x^2 = \|\vec{a}\|^2 = \|\vec{b}\|^2 = 2v^2 + y^2$. By multiplying with their common denominator, it is no loss of generality to presume $u, v, x, y \in \mathbb{Z}$. Now e) raises a contradiction. \square

In order to avoid the number-theoretic hassles of \mathbb{Q} leading to Lemma 38e+f) and the drowning of an early Greek mathematician, we now restrict to fields supporting the normalization of vectors:

Convention 39. *In the sequel, unless indicated otherwise, let $\mathbb{F} \subseteq \mathbb{C}$ be a field containing \sqrt{t} for each $0 < t \in \mathbb{F} \cap \mathbb{R}$.*

Observe that such \mathbb{F} must be of unbounded algebraic degree. In fact the set $\mathbb{A} \cap \mathbb{R}$ of algebraic reals satisfies this condition; but, by the **Abel-Ruffini Theorem**, smaller fields do as well. Based on Convention 39, we may always choose $\lambda = 1 = \|\vec{v}_i\|^2 = \|\vec{w}_i\|^2$ in Lemma 38 and thus have a sufficiently rich fund $\mathcal{U}(\mathbb{F}^d)$ of unitary matrices at our disposal for the following

- Lemma 40.** *a) Let $X_1, \dots, X_k \in \text{Gr}(\mathcal{H})$ be pairwise orthogonal and similarly for $Y_1, \dots, Y_k \in \text{Gr}(\mathcal{H})$ with $\dim(X_i) = \dim(Y_i)$. Then there exists $U \in \mathcal{U}(\mathcal{H})$ with $Y_i = U \cdot X_i$, $i = 1, \dots, k$.*
b) Let $Z \subseteq \text{Gr}(\mathcal{H})$ have $\dim(Z) =: d$. Then $\text{Gr}(Z)$ and $\text{Gr}(\mathbb{F}^d)$ are isomorphic.
c) Let $f(X_1, \dots, X_n)$ and $g(Y_1, \dots, Y_m)$ be formulas. Now consider the $(n+m)$ -variate formula $f+g := f(\vec{X}) \vee g(\vec{Y})$. Then $\text{maxdim}_{\mathbb{F}}(f+g, d) = \min \{d, \text{maxdim}_{\mathbb{F}}(f, d) + \text{maxdim}_{\mathbb{F}}(g, d)\}$. Moreover, for $k \in \mathbb{N}$, let $k \cdot f$ denote the $(k \cdot n)$ -variate formula $f + f + \dots + f$ (k -times). Then it holds $\text{maxdim}_{\mathbb{F}}(k \cdot f, d) = \min \{d, k \cdot \text{maxdim}_{\mathbb{F}}(f, d)\}$.
d) $\text{Gr}(\mathbb{F}^d) \times \text{Gr}(\mathbb{F}^e) \ni (X, Y) \mapsto X \oplus Y \in \text{Gr}(\mathbb{F}^{d+e})$ constitutes an ortho-embedding. For each formula f and $k, d \in \mathbb{N}$, it holds $\text{maxdim}_{\mathbb{F}}(f, d+k) \geq \text{maxdim}_{\mathbb{F}}(f, d) + \text{maxdim}_{\mathbb{F}}(f, k)$. In particular, if f is satisfiable over $\text{Gr}(\mathbb{F}^d)$ and over $\text{Gr}(\mathbb{F}^k)$, then so is it over $\text{Gr}(\mathbb{F}^{d+k})$.
e) For $I \subseteq \{1, \dots, n\} =: [n]$ abbreviate $g_I(Y_1, \dots, Y_n) := \bigwedge_{i \in I} Y_i \wedge \bigwedge_{i \notin I} \neg Y_i$, a formula of length $\mathcal{O}(n)$. Then, for distinct $I, J \subseteq [n]$, $g_I(\vec{Y}) \wedge g_J(\vec{Y}) = 0$. Now let $I_1, \dots, I_M \subseteq [n]$ be pairwise distinct and Z_1, \dots, Z_M pairwise orthogonal. Then there exist Y_1, \dots, Y_n with $g_{I_m}(\vec{Y}) \supseteq Z_m$ for all $m = 1, \dots, M$.
f) Let $k, m \in \mathbb{N}$. There exists a formula $\psi_{k,m}$, of length $\mathcal{O}(k \cdot m \cdot \log m)$ and independent of \mathbb{F} , with $\text{maxdim}_{\mathbb{F}}(\psi_{k,m}, d) = k \cdot \lfloor d/m \rfloor$.

The formula $\psi_{k,d}$ from Item g) may be considered as a projection functor (not to be confused with a projection mapping from Section 1.1.1) in that it maps $\text{Gr}(\mathbb{F}^d)$ onto $\text{Gr}_{\leq k}(\mathbb{F}^d)$ — which in turn is not to be confused with $\text{Gr}(\mathbb{F}^k)$. The formula $\neg \psi_{d,d}$ generalizes Example 7d) and strengthens Fact 37d); compare [Hagg09, SLIDE #19].

Proof (Lemma 40).

- a) Choose orthonormal bases for each X_i and combine and extend them to one $(\vec{x}_i)_{i \in I}$ for the entire space; similarly $(\vec{y}_i)_{i \in I}$ for the Y_i . Then the claim follows from Lemma 38d).
b) Decompose $\mathcal{H} = \mathbb{F}^d \times \mathcal{H}'$. By Lemma 36, $\text{Gr}(\mathbb{F}^d)$ is isomorphic to $\text{Gr}(\mathbb{F}^d \times 0)$; and according to Item a), the latter is unitarily equivalent (and hence isomorphic by virtue of Lemma 38c) to $\text{Gr}(Z)$.
c) It suffices to prove the first claim. Let $s := \text{maxdim}_{\mathbb{F}}(f, d)$ and $t := \text{maxdim}_{\mathbb{F}}(g, d)$. Then $\dim(V \vee W) \leq \dim(V) + \dim(W)$ implies $\dim(f(\vec{X}) \vee g(\vec{Y})) \leq \dim(f(\vec{X})) + \dim(g(\vec{Y})) \leq s + t$ for each $X_1, \dots, Y_m \in \text{Gr}(\mathbb{F}^d)$. Conversely, there exist $X_1, \dots, X_n, Y_1, \dots, Y_m \in \text{Gr}(\mathbb{F}^d)$ such that $V := f(\vec{X})$ and $W := g(\vec{Y})$ have $\dim(V) = s$ and $\dim(W) = t$. Now if $V \wedge W = 0$, we are done: $f(\vec{X}) \vee g(\vec{Y}) = V \vee W$ has $\dim(V \vee W) = s + t$. Otherwise consider some t -dimensional subspace $\tilde{W} \subseteq \mathbb{F}^d$ with $\dim(\tilde{W} \wedge V) = \max(0, s + t - d)$; for instance extend some orthogonal basis $\vec{v}_1, \dots, \vec{v}_s$ of V to one of \mathbb{F}^d and let \tilde{W} be spanned by the last t of $\vec{v}_1, \dots, \vec{v}_s, \vec{v}_{s+1}, \dots, \vec{v}_d$. Now according to a), there exists a unitary matrix U such that $\tilde{W} = U \cdot W$; and, by Lemma 38c), $\tilde{W} = g(U \cdot \vec{Y})$: hence $f(\vec{X}) \vee g(U \cdot \vec{Y}) = V \vee \tilde{W}$ has $\dim(V \vee \tilde{W}) = s + t - \max(0, s + t - d) = \min(d, s + t)$.

- d) Let $(X_1, \dots, X_N) \in \text{Gr}(\mathbb{F}^d)$ be such that $\dim(f(\vec{X})) = \text{maxdim}(f, d)$; and $(Y_1, \dots, Y_N) \in \text{Gr}(\mathbb{F}^k)$ such that $\dim(f(\vec{Y})) = \text{maxdim}(f, k)$. Now consider $Z_i := X_i \times Y_i \in \text{Gr}(\mathbb{F}^{d+k})$ and note that Z_i commutes with $\mathbb{F}^d \times 0^k$. By Lemma 36, $\Xi_{\mathbb{F}^{d+k}}(f; \vec{Z}) \wedge (\mathbb{F}^d \times 0^k) = \Xi_{\mathbb{F}^d}(f; \vec{X}) \times 0^k = \mathbb{F}^d \times 0^k$ has dimension $\text{maxdim}(f, d)$; and, symmetrically, $\Xi_{\mathbb{F}^{d+k}}(f; \vec{Z}) \wedge (0^d \times \mathbb{F}^k)$ has dimension $\text{maxdim}(f, k)$; hence $\Xi_{\mathbb{F}^{d+k}}(f; \vec{Z})$ has dimension $\text{maxdim}(f, d) + \text{maxdim}(f, k)$.
- e) Since $I \neq J$, there exists $i \in I \setminus J$ or $i \in J \setminus I$. In either case, the intersection $g_I(\vec{Y}) \wedge g_J(\vec{Y})$ will contain both terms Y_i and $\neg Y_i$, hence must be 0.
- For the second claim, let $Y_i := \bigvee_{m:i \in I_m} Z_m$. Then obviously $Y_i \supseteq Z_\ell$ for $i \in I_\ell$; and by orthogonality, $\neg Z_\ell \subseteq Z_m$ for $m \neq \ell$ implies $\neg Z_\ell \subseteq Y_i$ for $i \notin I_\ell$; hence $g_{I_\ell}(\vec{Y}) \supseteq Z_\ell$.
- f) The case $k > 1$ can be reduced to that of $k = 1$ by letting $\psi_{k,m} := k \cdot \psi_{1,m}$ according to Item c). In order to construct $\psi_{1,m}$, consider the projection $f(X, Z) := \Pi(\Pi(X, Z), X)$ of X first to Z and then back to X . Then, as in the first part of Fact 37c), it follows $\dim(f(X, Z)) \leq \min\{\dim(Z), \dim(X)\}$. Moreover, for $\vec{z}_1, \dots, \vec{z}_k, \vec{w}_1, \dots, \vec{w}_k$ pairwise orthogonal with $Z = \text{lspan}(\vec{z}_1, \dots, \vec{z}_k)$, $X := \text{lspan}\{\vec{z}_1 + \vec{w}_1, \dots, \vec{z}_k + \vec{w}_k\}$ has $f(X, Z) = X$. Now let g_I and distinct $I_1, \dots, I_M \subseteq [n]$ as in e), i.e. for $n := \lceil \log_2 M \rceil$; and define $\psi_{1,M}(X, Y_1, \dots, Y_n)$ as the iterated sequence of projections of X first to $Z_1 := g_{I_1}(\vec{Y})$, then back to X , then to $Z_2 := g_{I_2}(\vec{Y})$, back to X , to $Z_3 := g_{I_3}(\vec{Y})$, and so on until $Z_M := g_{I_M}(\vec{Y})$. Then it holds $\dim(\psi_{1,M}(X, \vec{Y})) \leq \min\{\dim(X), \dim(g_{I_1}(\vec{Y})), \dots, \dim(g_{I_M}(\vec{Y}))\}$, which is $\leq d/M$ for any $X, Y_1, \dots, Y_n \in \text{Gr}(\mathbb{F}^d)$ because the $g_{I_m}(\vec{Y})$ are pairwise disjoint according to e): thus $\text{maxdim}(\psi_{1,M}(X, \vec{Y})) \leq \lfloor d/M \rfloor =: k$.
- Conversely let $Z_1, \dots, Z_M \in \text{Gr}(\mathbb{F}^d)$ be pairwise orthogonal of $\dim(Z_m) = k$. By e) there exist Y_1, \dots, Y_n with $g_{I_m}(\vec{Y}) \supseteq Z_m$. And for orthogonal bases $\{\vec{z}_{1,m}, \dots, \vec{z}_{k,m}\}$ of Z_m , $X := \text{lspan}\{\vec{z}_{1,1} + \dots + \vec{z}_{1,M}, \vec{z}_{2,1} + \dots + \vec{z}_{2,M}, \dots, \vec{z}_{k,1} + \dots + \vec{z}_{k,M}\}$ has $f(X, Z_m) = X$ for each m ; thus $\psi_{1,M}(X, \vec{Y}) = X$ and $\text{maxdim}(\psi_{1,M}(X, \vec{Y})) = k$.
- The length of $\psi_{k,M} = k \cdot \psi_{1,M}$ is k -times the length of $\psi_{1,M}$. Concerning the latter, notice that Z and X appear twice (once negated, once positive) and three times in f , respectively; hence induction shows X to appear $(2M + 1)$ -times and each Z_m twice in $\psi_{1,M}$, where $Z_m = g_{I_m}$ itself has length $\mathcal{O}(n)$. \square

As indicated before, (strong) satisfiability fails dimensional heredity.

- Example 41.** a) Consider the formula h from Fact 37a). Then the formula $h + h$ according to Lemma 40c) is satisfiable precisely in even dimensions.
- b) Formula $5 \cdot (h|_h)$ is satisfiable in all dimensions except for $\{1, 2, 3, 6, 7, 11\}$.
- c) For $n \in \mathbb{N}$, the following formula Ψ_{2^n} in $2n + 1$ variables has length $\mathcal{O}(n)$ and is satisfiable precisely in dimensions $2^n \cdot \mathbb{N}$:

$$X_{n+1} \wedge \bigwedge_{i=1}^n \left((X_i \vee \neg Y_i) \wedge (Y_i \vee \neg X_i) \wedge (\neg X_i \vee \neg Y_i) \wedge \left((X_{i+1} \wedge (X_i \vee Y_i)) \vee (\neg X_{i+1} \wedge \neg (X_i \vee Y_i)) \right) \right)$$

- d) More generally, there is a formula $\Psi_d(\vec{X})$ of length $\mathcal{O}(\log d)$ which is satisfiable precisely in dimensions $d \cdot \mathbb{N}$; and any satisfying assignment $\vec{X} \in \text{Gr}(\mathbb{F}^{k \cdot d})$ of Ψ_d has $\dim(X_1) = k$.
- e) Formula $f(\vec{Y})$ is weakly satisfiable over $\text{Gr}(\mathbb{F}^d)$ iff the formula “ $f(\vec{Y}) \wedge X_0 = X_1 \wedge \Psi_d(\vec{X})$ ” of length $2|f| + \mathcal{O}(\log d)$ is (strongly) satisfiable over $\text{Gr}(\mathbb{F}^d)$, where “ $Z = W$ ” abbreviates $(Z \wedge W) \vee (\neg Z \wedge \neg W)$ according to Example 7g).

Proof. Satisfiability of f in dimension d means $\text{maxdim}(f, d) = d$.

- a) now follows from $\maxdim(2 \cdot h, d) = 2 \cdot \lfloor d/2 \rfloor$.
b) Fact 37 and Lemma 40c) imply $\maxdim(5 \cdot (h|_h), d) = \max(d, 5 \cdot \lfloor d/4 \rfloor)$.
c) Note that $X_i \vee \neg Y_i = 1$ in $\text{Gr}(\mathbb{F}^d)$ implies $\dim(X_i) + \dim(\neg Y_i) \geq d$; hence the first two terms in the big conjunction require $\dim(X_i) = \dim(Y_i)$. The fourth term amounts to condition $X_{i+1} = X_i \vee Y_i$ according to Example 7g); hence $\dim(X_{i+1}) = \dim(X_i) + \dim(Y_i)$ because of $X_i \wedge Y_i = 0$ (third term). Concluding, any satisfying assignment $(X_1, Y_1, \dots, X_n, Y_n, X_{n+1})$ of Ψ_{2^n} has $\dim(X_{i+1}) = 2 \cdot \dim(X_i)$; compare the geometric illustration in Figure 3. Hence $2^n \cdot \dim(X_1) = \dim(X_{n+1}) = d$ by the very first term of Ψ_{2^n} . Conversely, the following is easily verified to constitute a satisfying assignment of Ψ_{2^n} :

$$X_1 := \mathbb{F} \times \{0\}, \quad Y_1 := \{(x, x) : x \in \mathbb{F}\}, \quad X_2 := \mathbb{F}^2 \times \{0^2\}, \quad Y_2 := \{(\vec{x}, \vec{x}) : \vec{x} \in \mathbb{F}^2\}, \quad \dots, \\ X_{i+1} := \mathbb{F}^{2^i} \times \{0^{2^i}\}, \quad Y_{i+1} := \{(\vec{x}, \vec{x}) : \vec{x} \in \mathbb{F}^{2^i}\}, \quad \dots, \quad X_{n+1} = \mathbb{F}^{2^n}$$

(understood as all embedded into \mathbb{F}^{2^n} by appending zeros as necessary).

- d) Note that c) covers the case $d = 2^n$; indeed, $2^n \cdot \dim(X_1) = \dim(X_{n+1}) = d \cdot k$ implies $\dim(X_1) = k$. For the general case, let $n := \lfloor \log_2 d \rfloor$ (i.e. the least $n \in \mathbb{N}$ such that $2^n \geq d$). Now add to Ψ_{2^n} variables $Z_1, W_1, Z_2, W_2, \dots, Z_{n+1}, W_{n+1}$ with conditions $\dim(Z_i) = \dim(X_i)$ and $0 = W_1$ and $0 = Z_i \wedge W_i$ and $W_{i+1} = Z_i \vee W_i$: an overall length of $\mathcal{O}(n)$. So any satisfying assignment $\vec{X}, \vec{Y}, \vec{Z}, \vec{W}$ has

$$\dim\left(\bigvee_{i \in I} Z_{i+1}\right) = \sum_{i \in I} \dim(Z_{i+1}) = \sum_{i \in I} 2^i \cdot \dim(X_1)$$

for every $I \subseteq \{0, 1, \dots, n\}$. Hence choose I such that $d = \sum_{i \in I} 2^i$, the binary expansion of d and replace condition “ $1 = X_{n+1}$ ” (first term of Ψ_{2^n}) with “ $1 = \bigvee_{i \in I} Z_{i+1}$ ”.

- e) By d), a satisfying assignment $\vec{X}, \vec{Y} \in \text{Gr}(\mathbb{F}^d)$ has $\dim(X_1) = 1$; hence $f(\vec{Y}) \neq 0$ is weakly satisfied by \vec{Y} . Conversely, a weakly satisfying assignment \vec{Y} of f has $\dim(f(\vec{Y}) \wedge X_0) = 1$ for an appropriate X_0 ; and by Lemma 40a+b), this $X_1 := f(\vec{Y}) \wedge X_0$ can be extended to a satisfying assignment \vec{X} of Ψ_d . \square

We close this subsection with an observation concerning the relation between real and complex satisfiability:

Proposition 42. *Let $\mathbb{F} \subseteq \mathbb{R}$ denote a real field.*

- a) $\mathbb{F} + i\mathbb{F} \subseteq \mathbb{C}$ is a field. Conversely for a field $\mathbb{E} \subseteq \mathbb{C}$, $\mathbb{F} := \mathbb{R} \cap \mathbb{E}$ is a field (in general properly) contained in $\text{Re}(\mathbb{E}) := \{\text{Re}(x) : x \in \mathbb{E}\}$.
b) Let formula f be [weakly] satisfiable over $\text{Gr}(\mathbb{F}^d)$. Then it is also [weakly] satisfiable over $\text{Gr}((\mathbb{F} + i\mathbb{F})^d)$. More precisely, there is an embedding $\hat{\cdot} : \text{Gr}(\mathbb{F}^d) \rightarrow \text{Gr}((\mathbb{F} + i\mathbb{F})^d)$.
c) More precisely the mapping

$$\text{Gr}_k(\mathbb{F}^d) \ni X \mapsto \underline{X} := X + iX := \{\vec{v} + i\vec{w} : \vec{v}, \vec{w} \in X\} \in \text{Gr}_k((\mathbb{F} + i\mathbb{F})^d)$$

is well-defined and satisfies $\neg \underline{X} = \underline{\neg X}$ and $\underline{X \wedge Y} = \underline{X} \wedge \underline{Y}$ and $\underline{X \vee Y} = \underline{X} \vee \underline{Y}$.

- d) Let formula f be [weakly] satisfiable over $\text{Gr}((\mathbb{F} + i\mathbb{F})^d)$. Then it is also [weakly] satisfiable over $\text{Gr}(\mathbb{F}^{2d})$.
e) More precisely, the mapping

$$\text{Gr}_k((\mathbb{F} + i\mathbb{F})^d) \ni X \mapsto \tilde{X} := \{(\vec{v}, \vec{w}) : \vec{v}, \vec{w} \in \mathbb{F}^d, \vec{v} + i\vec{w} \in X\} \in \text{Gr}_{2k}(\mathbb{F}^{2d})$$

is well-defined and satisfies $\neg \tilde{X} = \widetilde{\neg X}$ and $\widetilde{(X \wedge Y)} = \tilde{X} \wedge \tilde{Y}$ and $\widetilde{(X \vee Y)} = \tilde{X} \vee \tilde{Y}$.

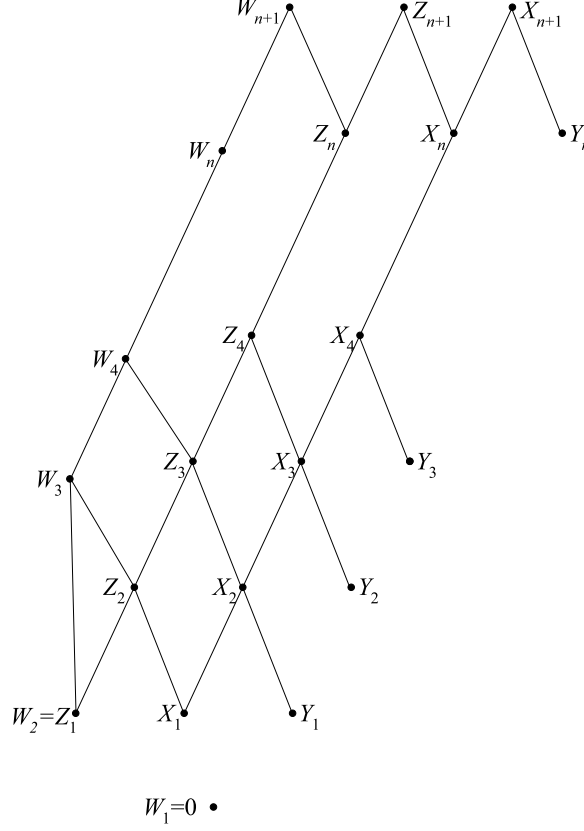


Fig. 3. Illustration to the proof of Example 41c+d).

Proof. a) Let $\mathbb{E} := \mathbb{Q}(i + \pi)$. Then $\mathbb{R} \cap \mathbb{E} = \mathbb{Q} \subsetneq \mathbb{Q}(\pi) = \text{Re}(\mathbb{E})$.

We record that, for $\vec{x}, \vec{y}, \vec{v}, \vec{w} \in \mathbb{F}^d$,

$$\langle \vec{x} + i\vec{y}, \vec{v} + i\vec{w} \rangle = \langle \vec{x}, \vec{v} \rangle + \langle \vec{y}, \vec{w} \rangle + i(\langle \vec{y}, \vec{v} \rangle - \langle \vec{x}, \vec{w} \rangle). \quad (12)$$

b) follows from c).

c) $(X + iX) \wedge (Y + iY) = (X \wedge Y) + i(X \wedge Y)$ is clear. Concerning $\neg \underline{X} = \underline{\neg X}$, calculate

$$\begin{aligned} \vec{x} + i\vec{y} \in \neg \underline{X} &\Leftrightarrow \langle \vec{x}, \vec{v} \rangle = 0 = \langle \vec{y}, \vec{w} \rangle \quad \forall \vec{v}, \vec{w} \in X \\ &\stackrel{(*)}{\Leftrightarrow} 0 = \langle \vec{x}, \vec{a} \rangle + \langle \vec{y}, \vec{b} \rangle + i\langle \vec{y}, \vec{a} \rangle - i\langle \vec{x}, \vec{b} \rangle \stackrel{(12)}{=} \langle \vec{x} + i\vec{y}, \vec{a} + i\vec{b} \rangle \quad \forall \vec{a}, \vec{b} \in X \\ &\Leftrightarrow \vec{x} + i\vec{y} \in \neg \underline{X} \end{aligned}$$

where the two implications in $(*)$ can be seen to hold by choosing $(\vec{x}, \vec{y}) := (\vec{a}, \vec{b})$ respectively (\vec{b}, \vec{a}) and $(\vec{a}, \vec{b}) := (\vec{x}, 0)$ respectively $(0, \vec{y})$.

d) follows from e).

e) Notice that, obviously, $\widetilde{X \wedge Y} = \widetilde{X} \wedge \widetilde{Y}$. Also, linear in-/dependence over $\mathbb{F} + i\mathbb{F}$ of k vectors $\vec{v}_1 + i\vec{w}_1, \dots, \vec{v}_k + i\vec{w}_k \in (\mathbb{F} + i\mathbb{F})^d$ is equivalent to the linear in-/dependence over

\mathbb{F} of the $2k$ vectors $(\vec{v}_1), \dots, (\vec{v}_k), (-\vec{v}_1), \dots, (-\vec{v}_k) \in \mathbb{F}^{2d}$.

$$\begin{aligned} 0 &\stackrel{!}{=} \sum_{j=1}^k (\lambda_j - i\mu_j)(\vec{v}_j + i\vec{w}_j) = \sum_j \lambda_j \vec{v}_j + \sum_j \mu_j \vec{w}_j + i \sum_j \lambda_j \vec{w}_j - i \sum_j \mu_j \vec{v}_j \\ &\Leftrightarrow 0 = \sum_{j=1}^k \lambda_j \begin{pmatrix} \vec{v}_j \\ \vec{w}_j \end{pmatrix} + \sum_{j=1}^k \mu_j \begin{pmatrix} \vec{w}_j \\ -\vec{v}_j \end{pmatrix}. \end{aligned}$$

Since $X \in \text{Gr}((\mathbb{F} + i\mathbb{F})^d)$ is closed under scaling by i , $\vec{v} + i\vec{w} \in X$ implies $\vec{w} - i\vec{v} = -i(\vec{v} + i\vec{w}) \in X$. We thus see that $X = \text{lspan}_{\mathbb{F} + i\mathbb{F}}(\vec{v}_1 + i\vec{w}_1, \dots, \vec{v}_k + i\vec{w}_k)$ with $k = \dim_{\mathbb{F} + i\mathbb{F}}(X)$ has $\tilde{X} = \text{lspan}_{\mathbb{F}}((\vec{v}_1), \dots, (\vec{v}_k), (-\vec{v}_1), \dots, (-\vec{v}_k))$ and $\dim_{\mathbb{F}}(\tilde{X}) = 2k$. Now, by definition,

$$\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} \in \neg \tilde{X} \quad \Leftrightarrow \quad 0 = \langle \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}, \begin{pmatrix} \vec{v} \\ \vec{w} \end{pmatrix} \rangle = \langle \vec{x}, \vec{v} \rangle + \langle \vec{y}, \vec{w} \rangle \quad \forall \vec{v} + i\vec{w} \in X$$

and, proceeding to $\vec{w} - i\vec{v} \in X$, also $0 = \langle \vec{x}, \vec{w} \rangle - \langle \vec{y}, \vec{v} \rangle$. That in turn is equivalent to the vanishing of both real and imaginary part of $\langle \vec{x} + i\vec{y}, \vec{v} + i\vec{w} \rangle$ according to Equation (12) and for all $\vec{v} + i\vec{w} \in X$: that is, equivalent to $\vec{x} + i\vec{y} \in \neg X$ and thus to $\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} \in \neg \tilde{X}$. \square

4.2 Satisfiability is \mathcal{NP} -hard in any fixed Dimension

Consider dimensions $k < d$. It may first seem obvious that the d -dimensional satisfiability problem is computationally at least as hard as the k -dimensional one. On second thought by Lemma 36, a formula f may be logically easier to weakly satisfy over $\text{Gr}(\mathbb{F}^d)$ than over $\text{Gr}(\mathbb{F}^k)$; recall also Theorem 29. Moreover there is this important difference between satisfiability and weak satisfiability.

Theorem 43. Fix $1 \leq k < d$ and \mathbb{F} as in Convention 39.

- a) Then k -dimensional satisfiability is polynomial-time reducible[†] to d -dimensional satisfiability.
- b) And d -dimensional satisfiability is polynomial-time equivalent to d -dimensional weak satisfiability.

In particular, d -dimensional satisfiability and weak satisfiability are both \mathcal{NP} -hard.

Proof (Theorem 43). Recall that satisfiability of f over $\text{Gr}(\mathbb{F}^d)$ means $\text{maxdim}_{\mathbb{F}}(f, d) = d$; and weak satisfiability means $\text{maxdim}_{\mathbb{F}}(f, d) \geq 1$.

- b) Hence f is weakly satisfiable over $\text{Gr}(\mathbb{F}^d)$ iff $d \cdot f$ according to Lemma 40c) is satisfiable over $\text{Gr}(\mathbb{F}^d)$. Similarly, f is satisfiable over $\text{Gr}(\mathbb{F}^d)$ iff the restriction $\psi_{1,d}|_f$ according to Fact 37b) and Lemma 40f) is weakly satisfiable over $\text{Gr}(\mathbb{F}^d)$.
- a) Similarly observe that f is weakly satisfiable over $\text{Gr}(\mathbb{F}^k)$ iff the restriction $f_{\psi_{k,d}}$ according to Fact 37b) and Lemma 40f) is weakly satisfiable over $\text{Gr}(\mathbb{F}^d)$.

Note that all three formula transformations $f \mapsto d \cdot f$, and $f \mapsto \psi_{1,d}|_f$, and $f \mapsto f_{\psi_{k,d}}$ are easy to implement on a Turing machine in time polynomial in $(d, k$ and) the length of f . \square

[†] That is, there exists a Turing machine which, given a quantum logic formula f , can in polynomial time produce another formula g with the following property: f is satisfiable over $\text{Gr}(\mathbb{F}^k)$ iff g is satisfiable over $\text{Gr}(\mathbb{F}^d)$.

4.3 Satisfiability in Polynomial Space (and Better?)

As mentioned before, [DHMW05, COROLLARY 7] shows satisfiability over $\text{Gr}(\mathbb{C}^d)$ of a given formula f to be decidable; in fact uniformly in d . That is, a Turing machine can answer within finite time, given $d \in \mathbb{N}$ and (an encoding of) $f(X_1, \dots, X_n)$, whether f admits a [weakly] satisfying assignment $(W_1, \dots, W_n) \in \text{Gr}(\mathbb{C}^d)$. The argument proceeds by showing how to computationally transform the given (d, f) into a first-order sentence f^* over \mathbb{R} (not over \mathbb{C} , since orthogonality involves complex conjugation!) with coefficients from $\{-2, -1, 0, +1, +2\}$ which is true iff f is [weakly] satisfiable over $\text{Gr}(\mathbb{C}^d)$ [DHMW05, THEOREM 6]; and truth of f^* can be algorithmically decided by Tarski's Quantifier Elimination.

Concerning computational complexity, general quantifier elimination is known to be EXPSPACE-complete [MaMe82], i.e. provably unsolvable using asymptotically less than exponential space. On the other hand, more efficient algorithms are known for deciding first-order sentences over \mathbb{R} which avoid iterative elimination of quantifiers. The best of them have a running time exponential in the number of variables and doubly exponential in the number of quantifier alternations; and memory consumption polynomial in the number of variables and single exponential in the number of quantifier alternations: cf. [Ren92a, Ren92b] and, e.g., [CuGr97]. Now the first-order sentence f^* constructed in [DHMW05] employs both universal and existential quantifiers; and we observe that one can get rid of one kind in Item b) of the following

Theorem 44. *Fix $d \in \mathbb{N}$ and let \mathbb{F} denote either \mathbb{C} or \mathbb{R} . Moreover write \mathbb{A} for the set of algebraic reals and $\mathbb{A} + i\mathbb{A}$ for algebraic numbers.*

- a) *A quantum logic formula is [weakly] satisfiable over $\text{Gr}(\mathbb{R}^d)$ iff it is [weakly] satisfiable over $\text{Gr}(\mathbb{A}^d)$; it is [weakly] satisfiable over $\text{Gr}(\mathbb{C}^d)$ iff it is [weakly] satisfiable over $\text{Gr}((\mathbb{A} + i\mathbb{A})^d)$.*
- b) *There exists a nondeterministic BSS machine devoid of constants which, upon input of a quantum logic formula f , within a number of steps polynomial in d and in the length of f , accepts iff f admits a [weakly] satisfying assignment over $\text{Gr}(\mathbb{F}^d)$.*
- c) *Upon input of d and f as above, a Turing machine can within time polynomial in d and in the length of f construct a quartic polynomial $p_{f,d}$ with coefficients from $\{0, \pm 1, \pm 2\}$ having the following property: f admits a [weakly] satisfying assignment over $\text{Gr}(\mathbb{F}^d)$ iff $p_{f,d}$ has a real root.*
- d) *[Weak] satisfiability over $\text{Gr}(\mathbb{F}^d)$ of a given formula f is decidable by a Turing machine in space polynomial in d and the length of f , i.e. belongs to the complexity class PSPACE.*
- e) *The function $(f, d) \mapsto \max\dim_{\mathbb{F}}(f, d)$ is computable in space polynomial in d and the length of f . More precisely, given integers d, k and formula f , a nondeterministic BSS machine without constants can decide in time polynomial in d and the length of f whether $\max\dim_{\mathbb{F}}(f, d) \geq k$ holds.*

A nondeterministic BSS machine is a BSS machine with the additional capability of ‘guessing’ real numbers, just like a nondeterministic Turing machine may guess bits. Nondeterministic polynomial-time BSS decidability admits a well-known characterization:

Fact 45. *Fix $\mathbb{F}, \mathbb{E} \subseteq \mathbb{C}$ and consider the following decision problem:*

$$\text{Given some polynomial } p \in \mathbb{E}[X_1, \dots, X_n], \text{ does it admit a root in } \mathbb{F} ? \quad (13)$$

where a root of p in \mathbb{F} is some $\vec{x} \in \mathbb{F}^n$ such that $p(\vec{x}) = 0$.

- a) For $\mathbb{E} = \mathbb{R} = \mathbb{F}$ and to the BSS-machine model, this computational problem is $\mathcal{NP}_{\mathbb{R}}$ -complete.

More precisely, a polynomial-time Turing (!) machine can, given the description of a BSS machine \mathcal{M} with symbolic names for its real constants and given an input $\vec{y} \in \mathbb{R}^m$, output the dense encoding (coefficient list of its $\mathcal{O}(m^4)$ monomials) of a multivariate polynomial $p_{\mathcal{M}}(X_1, \dots, X_N; y_1, \dots, y_m)$ of total degree at most 4 such that the following holds: \mathcal{M} accepts \vec{y} iff $p_{\mathcal{M}}(\cdot; \vec{y})$ admits a real root.

- b) For nondeterministic BSS-machines without constants and on binary inputs, the above problem (13) is complete in the same sense when restricted to polynomials $p_i(X_1, \dots, X_N)$ of degree at most 4 and with coefficients in $\mathbb{E} := \{0, \pm 1, \pm 2\}$ (but no symbolic parameters $y_i \in \mathbb{R} =: \mathbb{F}$).
- c) In fact in the ordered case $\mathbb{F} \subseteq \mathbb{R}$, the feasibility of a system

$$p_1 = \dots = p_k = 0, \quad q_1 \neq 0, \dots, q_\ell \neq 0 \quad (14)$$

of polynomial in/equalities with coefficients from $\mathbb{E} := \mathbb{N}$ can be reformulated (by a Turing machine in time polynomial in the binary description length of the list of nonzero monomials and their coefficients) as the feasibility a single equation $p(\vec{X}) = 0$ with p_0 a sum of squares of quadratic polynomials, each with coefficients from $\{0, \pm 1\}$.

- d) For $\mathbb{F} = \mathbb{R}$, the feasibility of Equation (13) can be decided by a Turing machine in time polynomial in the description length of the input polynomial [Grig88, HRS90, Cann88, MaTo97].
- e) For $\mathbb{F} = \mathbb{N}$, the feasibility of (13) is undecidable to a Turing machine [Mati70].
- f) For $\mathbb{F} = \mathbb{Q}$, the decidability of (13) is open; cf. e.g. [Poon09].
- g) For $\mathbb{F} = \mathbb{C}$, the answer to (13) is trivially positive unless $p \equiv c \neq 0$. However, the feasibility of Equation (14) is non-trivial; yet can, subject to the **Generalized Riemann Hypothesis**, be decided in $\text{coRP}^{\mathcal{NP}}$ (which is a complexity class included in, and believed much smaller than, PSPACE): cf. [Koir96].

We shall refer to (13) with $\mathbb{E} = \{0, \pm 1, \pm 2\}$ and $\mathbb{F} = \mathbb{R}$ as *Feasibility of Real Polynomials* [BCSS98, SECTION 1.2.6].

Item a) is well-known; see e.g. [BSS89, BCSS98, MeMi97]. For the additional claims on the coefficients, observe that computation nodes may be restricted to the arithmetic primitives $+$, $-$, \times , \div and machine constants y_i as unary operations. Similarly, integer coefficients entering from the next node map $\beta(m)$ can be replaced by an addition chain over 1 of length $\mathcal{O}(\log \beta(m))$. Item b) is established similarly.

Proof (Theorem 44).

- a) The proof of [DHMW05, THEOREM 6] shows how to reformulate satisfiability of f over $\text{Gr}(\mathbb{R}^d)$ as real feasibility of a system of polynomial equations with coefficients from \mathbb{Z} . Now by the Tarski-Seidenberg Principle [BPR03, THEOREM 2.80], such a system is feasible iff it is feasible over \mathbb{A} . The case of $\text{Gr}(\mathbb{C}^d)$ can be treated similarly by separately considering the real and imaginary parts.
- b) A nondeterministic BSS machine may guess a satisfying assignment for the variables of f , that is a collection X_1, \dots, X_n of subspaces of $\text{Gr}(\mathbb{F}^d)$ represented by matrices $A_1, \dots, A_n \in \mathbb{F}^{d \times d}$ as $X_i = \text{range}(A_i)$. It can then evaluate $f(X_1, \dots, X_n)$ in polynomial time, invoking Gaussian Elimination (order d^3 steps) for each orthogonal complementation $X \mapsto \neg X$, intersection $(X, Y) \mapsto X \wedge Y$, or sum $(X, Y) \mapsto X \vee Y$; and accept iff the resulting matrix is non-zero/regular.

- c) Follows from b) by virtue of Fact 45c). (An alternative, more self-contained proof will be given in Theorem 72c)
- d) follows from b) and/or c) by virtue of Fact 45d).
- e) similarly. □

The interesting question is of course whether an upper complexity bound better than PSPACE is possible. However, “ \mathcal{NP} versus PSPACE” has turned out equally challenging as the millennium “ \mathcal{P} versus \mathcal{NP} ” [FoKo00].

- Remark 46.** a) Note that in Theorem 44b), a BSS machine \mathcal{M} over \mathbb{C} (rather than over \mathbb{R}) cannot even verify orthogonality of two vectors $\vec{x}, \vec{y} \in \mathbb{C}^2$: Restricted to purely complex arithmetic, \mathcal{M} has no ability to separate real and imaginary parts or to perform complex conjugation.
- b) There exists a vast literature on structural complexity theory for BSS machines [MeMi97, CuGr97]. Rewritten in their notation, Theorem 44b) means that [weak] d -dimensional satisfiability of a given quantum logic formula belongs to the class $\text{BP}(\mathcal{NP}_{\mathbb{R}}^0)$.
- c) This is the second-lowest level (lowest being $\text{BP}(\mathcal{P}_{\mathbb{R}}^0)$) of the so-called BSS polynomial hierarchy $\text{BP}(\text{PH}_{\mathbb{R}}^0)$ [Cuck93], all contained within (Turing-) PSPACE.

It is therefore rather unlikely that real polynomial feasibility (and thus quantum satisfiability) be PSPACE-hard.

4.4 Ring Embedding à la von Staudt/von Neumann

Already JOHN VON NEUMANN had exploited a construction he credited to VON STAUDT for embedding the ring $(\mathbb{F}, 0, 1, +, -, \times)$ into 3D quantum logic and, more generally, the matrix ring $\mathbb{F}^{d \times d}$ into $\text{Gr}(\mathbb{F}^{3d})$ [Neum60].

Example 47. Let $\mathbb{F} \subseteq \mathbb{C}$ denote any field and fix $d \in \mathbb{N}$.

- a) For $T \in \mathbb{F}^{d \times e}$ write $\text{graph}(T) := \{(\vec{x}, T \cdot \vec{x}) : \vec{x} \in \mathbb{F}^d\}$; cmp. Figure 4. Then $\text{graph}(T) \in \text{Gr}_d(\mathbb{F}^{d+e})$ has $0 = \text{graph}(T) \wedge \neg \text{graph}(0)$ and $\neg \text{graph}(T) = \{(-T^\dagger \cdot \vec{y}, \vec{y}) : \vec{y} \in \mathbb{F}^e\}$ and $1 = \text{graph}(T) \vee \neg \text{graph}(0)$. Here $\text{graph}(0) = \mathbb{F}^d \times 0^e$ and $T^\dagger \in \mathbb{F}^{e \times d}$ denotes the adjoint (conjugate transpose) of T . Moreover, in case $\text{rank}(T) = d$, it holds $0 = \text{graph}(T) \wedge \text{graph}(0)$ and $\neg \text{graph}(T) = \text{graph}(-T^{\dagger-1}) = \text{graph}(-T^{-1\dagger})$.
- b) Conversely let $Y \in \text{Gr}(\mathbb{F}^{d+e})$ satisfy $0 = Y \wedge \neg \text{graph}(0)$ and $1 = Y \vee \neg \text{graph}(0)$. Then there exists a unique $T \in \mathbb{F}^{d \times e}$ with $Y = \text{graph}(T)$. And from $\text{graph}(T) \wedge \text{graph}(0) = 0$ it follows $\text{rank}(T) = d$.
- c) Slightly more generally, for $T, S, R \in \mathbb{F}^{d \times d}$, write

$$\begin{aligned}
 X_0(T) &:= \{(0, \vec{x}, -T \cdot \vec{x}) : \vec{x} \in \mathbb{F}^d\} & X^0(T) &:= \{(0, -T \cdot \vec{x}, \vec{x}) : \vec{x} \in \mathbb{F}^d\} \\
 X_1(S) &:= \{(-S \cdot \vec{x}, 0, \vec{x}) : \vec{x} \in \mathbb{F}^d\} & X^1(S) &:= \{(\vec{x}, 0, -S \cdot \vec{x}) : \vec{x} \in \mathbb{F}^d\} \\
 X_2(R) &:= \{(\vec{x}, -R \cdot \vec{x}, 0) : \vec{x} \in \mathbb{F}^d\} & X^2(R) &:= \{(-R \cdot \vec{x}, \vec{x}, 0) : \vec{x} \in \mathbb{F}^d\} .
 \end{aligned}$$

- i) For each fixed $j \in \mathbb{Z}_3 = \{0, 1, 2\}$ and $T \in \mathbb{F}^{d \times d}$, $X := X_j(T)$ satisfies $X \wedge \neg X_j(0) = 0$ and $X \vee \neg X_j(0) = \mathbb{F}^{3d}$; $X \wedge X_j(0) = 0$ if $T \in \text{GL}(\mathbb{F}^d)$.

- ii) Conversely, any $X \in \text{Gr}(\mathbb{F}^{3d})$ with $X \wedge \neg X_j(0) = 0$ and $X \vee \neg X_j(0) = 1$ has the form $X = X_j(T)$ for a unique $T \in \mathbb{F}^{d \times d}$; $T \in \text{GL}(\mathbb{F}^d)$ if $X \wedge X_j(0) = 0$.
 iii) + iv) Similarly to i) + ii) but for $X^j(T)$ instead of $X_j(T)$.
 d) It holds

$$X^j(0) = X_{j+1}(0), \quad \dim(X_j) = d, \quad \text{and } X_j(0) \perp X_i(0) \text{ for } i \neq j. \quad (15)$$

Conversely, to any $X_0, X_1, X_2, X^0, X^1, X^2 \in \text{Gr}(\mathbb{F}^{3d})$ satisfying Equation (15) and under the additional hypothesis of Convention 39, there exists unitary $U \in \mathbb{F}^{3d \times 3d}$ with $X_j = U \cdot X_j(0)$ and $X^j = U \cdot X^j(0)$ simultaneously for $j = 0, 1, 2$.

- e) For all $j \in \mathbb{Z}_3$ it holds

$$\begin{aligned} (X_j(T) \vee X_{j+1}(S)) \wedge \neg X^j(0) &= X^{j+2}(S \cdot T) \quad \text{and} \\ (X^j(T) \vee X^{j+1}(S)) \wedge \neg X^j(0) &= X_{j+2}(T \cdot S). \end{aligned} \quad (16)$$

- f) It holds $\neg X_j(T) \wedge \neg X^{j+1}(0) = X^j(-T^\dagger)$ and, for $T \in \text{GL}(\mathbb{F}^d)$, $X_j(T) = X^j(T^{-1})$.
 g) $\left(\left((X_j(T) \vee X^{j+1}(S)) \wedge (X^{j+2}(\text{id}) \vee X_{j+1}(0)) \right) \vee X_{j+2}(0) \right) \wedge \neg X_{j+2}(0) = X_j(T - S)$.
 h) Fix $R_0, R_1, R_2 \in \text{GL}(\mathbb{F}^d)$ and consider $\tilde{X}_j(T) := X_j(R_j \cdot T \cdot R_{j-1}^{-1})$ and $\tilde{X}^j(T) := X^j(R_{j-1} \cdot T \cdot R_j^{-1})$. Then it holds $\tilde{X}_j(0) = X_j(0)$ as well as Equation (16) and Item g) with \tilde{X}_j, \tilde{X}^j instead of X_j, X^j . Moreover $\tilde{X}_j(T) = \tilde{X}^j(T^{-1})$ in case $T \in \text{GL}(\mathbb{F}^d)$. In particular $\tilde{X}_j(\text{id}) = X_j(W_j) = X^j(W_j^{-1}) = \tilde{X}^j(\text{id})$ for $W_j := R_j \cdot R_{j-1}^{-1}$ and $W_{j+2} \cdot W_{j+1} \cdot W_j = 1$.
 j) Conversely suppose $\tilde{X}_j = \tilde{X}^j = X_j(W_j) = X_j(W_j^{-1})$ satisfy $(\tilde{X}_j \vee \tilde{X}_{j+1}) \wedge \neg X^j(0) = \tilde{X}^{j+2}$ for $j = 0, 1, 2$. Then there exist $R_0, R_1, R_2 \in \text{GL}(\mathbb{F}^d)$ with $W_j = R_j \cdot R_{j-1}^{-1}$.

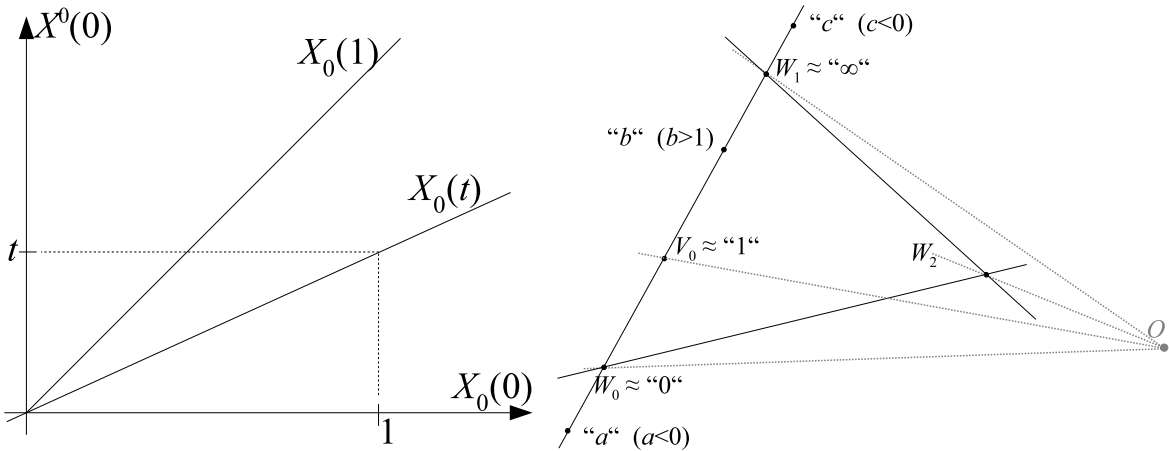


Fig. 4. Left: Encoding \mathbb{F} as slopes in $\text{Gr}(\mathbb{F}^2)$. Right: A 3-frame in 3D coordinatize scalar arithmetic operations.

Proof. a) The first claims are readily verified. Observe that $\langle (\vec{x}, T\vec{x}), (-T^\dagger \vec{y}, \vec{y}) \rangle = -\langle \vec{x}, T^\dagger \vec{y} \rangle + \langle T\vec{x}, \vec{y} \rangle = 0$ by the very definition of the adjoint; hence $\{(-T^\dagger \vec{y}, \vec{y}) : \vec{y} \in \mathbb{F}^e\} \subseteq \neg \text{graph}(T)$; and, since the dimensions of both sides amount to e , they must coincide. Moreover, $(\vec{x}, T\vec{x}) \in \text{graph}(T) \wedge \text{graph}(0)$ requires $T\vec{x} = 0$, hence $\vec{x} = 0$ in case $\text{rank}(T) = d$.

- b) Uniqueness of T is obvious. Let $Y = \text{lspan}_{\mathbb{F}} \left(\begin{pmatrix} \vec{x}_1 \\ \vec{y}_1 \end{pmatrix}, \dots, \begin{pmatrix} \vec{x}_k \\ \vec{y}_k \end{pmatrix} \right)$ with $\begin{pmatrix} \vec{x}_i \\ \vec{y}_i \end{pmatrix} \in \mathbb{F}^d \oplus \mathbb{F}^e$ linearly independent. Then $0 = \sum_i \lambda_i \vec{x}_i$ implies $\sum_i \lambda_i \begin{pmatrix} \vec{x}_i \\ \vec{y}_i \end{pmatrix} \in \neg \text{graph}(0) \wedge Y = 0$ by hypothesis; hence linear independence of $\begin{pmatrix} \vec{x}_i \\ \vec{y}_i \end{pmatrix}$ requires $\lambda_i \equiv 0$, showing that already the \vec{x}_i are linearly independent. On the other hand, $1 = Y \vee \neg \text{graph}(0)$ implies that the \vec{x}_i span entire \mathbb{F}^d , hence form a basis and yield well-definition of the linear map $T : \mathbb{F}^d \rightarrow \mathbb{F}^e$ via $T(\vec{x}_i) := \vec{y}_i$. If additionally $\text{graph}(T) \wedge \text{graph}(0) = 0$, we conclude as above that also the \vec{y}_i are linearly independent, hence $\text{rank}(T) = \dim \text{lspan}(\vec{y}_1, \dots, \vec{y}_d) = d$.
- c) Similarly.
- d) The first claim is easily verified. For the converse, choose three orthonormal bases of X_0, X_1, X_2 . Pairwise orthogonality and $\dim(X_j) = d$ imply that, together, they form an orthonormal basis of \mathbb{F}^{3d} . Put into a $(3d) \times (3d)$ -matrix U , this will be unitary and map $X_j(0)$ to X_j ; compare Lemma 40b).
- e) We consider the case $j = 0$ and $(0 - S \cdot \vec{y}, \vec{x} + 0, \vec{y} - T \cdot \vec{x}) \in X_0(T) + X_1(S)$. This belongs to $\neg X^0(0) = \mathbb{F}^d \times \mathbb{F}^d \times 0^d$ iff $\vec{y} = T \cdot \vec{x}$ and, in this case, evaluates to $(-S \cdot T \cdot \vec{x}, \vec{x}, 0) \in X^2(S \cdot T)$. The other cases proceed similarly.
- f) By the very definition of scalar product and adjoint, $\langle (0, \vec{x}, -T \cdot \vec{x}), (0, \vec{y}, \vec{z}) \rangle = \langle \vec{x}, \vec{y} \rangle - \langle \vec{x}, T^\dagger \cdot \vec{z} \rangle$ vanishes for all $\vec{x} \in \mathbb{F}^d$ iff $\vec{y} = T^\dagger \vec{z}$.
- g) Straight-forward calculation similarly to e) but more tedious.
- h) even more tedious.
- j) The prerequisite implies $X^{j+2}(W_{j+2}^{-1}) = \tilde{X}^{j+2} = (\tilde{X}_j \vee \tilde{X}_{j+1}) \wedge \neg X^j(0)$ which furtheron coincides with $(X_j(W_j) \vee X_{j+1}(W_{j+1})) \wedge \neg X^j(0) = X^{j+2}(W_{j+1} \cdot W_j)$ by e). Hence $W_{j+2} \cdot W_{j+1} \cdot W_j = 1$ by c). Now let, say, $R_0 := W_0$, $R_2 := \text{id}$, and $R_1 := W_1 \cdot R_0 = W_2^{-1}$. \square

Note how Item c) encodes the elements of the matrix ring $\mathbb{F}^{d \times d}$ into $\text{Gr}(\mathbb{F}^{3d})$ such that (Items e-g) the ring operations “ \times ”, “ $-$ ” (and thus also “ $+$ ”) and “ \dagger ” can be expressed as quantum logic formulas. The encoding is relative to $X_j(0)$, $X^j(0)$, $X_j(\text{id})$, and $X^j(\text{id})$. However, by virtue of Example 47d+f), $X^0(T)$, $X^1(T)$, $X^2(T)$ and $X_1(T)$, $X_2(T)$ can be expressed as a quantum logic function of $X_0(0)$, $X_1(0)$, $X_2(0)$, $X_0(\text{id})$, $X_1(\text{id})$, $X_2(\text{id})$, and $X_0(T)$ only. For instance, $X_1(T) = X_1(T \cdot \text{id}) = (X^2(T) + X^0(\text{id})) \wedge \neg X^0(0)$ and $X^2(T) = X^2(T \cdot \text{id}) = (X_0(T) + X_1(\text{id})) \wedge \neg X^0(0)$; similarly for $X^1(T)$ and $X_2(T)$. The dependence thus boils down to the 5-tuple $(X_0(0), X_1(0), X_2(0), X_0(\text{id}), X_1(\text{id})) =: (W_0, W_1, W_2, V_0, V_1)$ constituting a normalized orthogonal (von Neumann) 3-frame [Neum60, DEFINITION II.5.2]:

$$W_0 \perp W_1 \perp W_2 \perp W_0 \ \&\& \ W_0 \vee V_0 = W_1 \vee V_0 = W_0 \vee W_1 \ \&\& \ W_0 \vee V_1 = W_2 \vee V_1 = W_0 \vee W_2. \quad (17)$$

Such a frame in a sense fixes a ‘coordinate system’ of \mathbb{F} within $\text{Gr}(\mathbb{F}^3)$; cmp. Figure 4: Items h+j) describe how the transition to another 3-frame $(W_0, W_1, W_2, \tilde{V}_0, \tilde{V}_1)$ in $\text{Gr}(\mathbb{F}^{3d})$ translates to a ‘scaling’ within $\mathbb{F}^{d \times d}$. And Item d) asserts that (W_0, W_1, W_2) can be defined uniquely up to unitary equivalence: provided that the underlying field \mathbb{F} supports normalization of vectors (Convention 39). Now Lemma 38f) has shown that, without this provision and in particular for the case $\mathbb{F} = \mathbb{Q}$, two orthogonal 3-frames (W_0, W_1, W_2) and $(\tilde{W}_0, \tilde{W}_1, \tilde{W}_2)$ need *not* be unitarily equivalent in general. On the other hand, VON NEUMANN succeeded in defining an embedding $\mathbb{F}^{d \times d} \rightarrow \text{Gr}(\mathbb{F}^{3d})$ as in Example 47c) in terms of an orthogonal 3-frame $(W_0, W_1, W_2, V_0, V_1)$ only [Neum60, §II.4-II.8]. In particular we record

Fact 48. *There exist quantum formulas*

$$f_{\times}(X, Y; W_0, W_1, W_2, V_0, V_1), \quad f_{-}(X, Y; W_0, W_1, W_2, V_0, V_1), \quad f_{\dagger}(X; W_0, W_1, W_2, V_0, V_1)$$

such that, for every field $\mathbb{F} \subseteq \mathbb{C}$ and every $d \in \mathbb{N}$ and every 5-tuple $W_0, W_1, W_2, V_0, V_1 \in \text{Gr}_d(\mathbb{F}^{3d})$ satisfying Equation (17), there exists an embedding $\Phi = \Phi_{W_0, W_1, W_2, V_0, V_1} : \mathbb{F}^{d \times d} \rightarrow \text{Gr}_d(\mathbb{F}^{3d})$ satisfying $\Phi(0) = W_0$ and $\Phi(\text{id}) = V_0$ and

$$\begin{aligned} f_{\times}(\Phi(A), \Phi(B); W_0, W_1, W_2, V_0, V_1) &= \Phi(A \cdot B), \\ f_{-}(\Phi(A), \Phi(B); W_0, W_1, W_2, V_0, V_1) &= \Phi(A - B), \\ f_{\dagger}(\Phi(A); W_0, W_1, W_2, V_0, V_1) &= \Phi(A^{\dagger}) . \end{aligned}$$

Recall that polynomials are terms over the structure of a commutative ring $(R, +, \times)$. Two extensions, $*$ -polynomials are terms over a ring with involution $(R, +, \times, *)$; and noncommutative polynomials are terms over a noncommutative ring structure.

Corollary 49. *Let $p(Y_1, \dots, Y_n)$ denote a noncommutative $*$ -polynomial with integer coefficients.*

- a) *There exists a quantum logic formula $f_p(X_1, \dots, X_n; W_0, W_1, W_2, V_0, V_1)$ such that, for all field $\mathbb{F} \subseteq \mathbb{C}$ and all $d \in \mathbb{N}$ and all $A_1, \dots, A_n \in \mathbb{F}^{d \times d}$ and every choice of W_0, W_1, W_2, V_0, V_1 satisfying Equation (17), it holds*

$$\begin{aligned} f_p(\Phi_{W_0, W_1, W_2, V_0, V_1}(A_1), \dots, \Phi_{W_0, W_1, W_2, V_0, V_1}(A_n); W_0, W_1, W_2, V_0, V_1) &= \\ &= \Phi_{W_0, W_1, W_2, V_0, V_1}(p(A_1, \dots, A_n)) . \end{aligned}$$

- b) *Furthermore there exists a quantum logic formula $g_p(X_1, \dots, X_n; W_0, W_1, W_2, V_0, V_1)$ whose satisfiability over $\text{Gr}(\mathbb{F}^{3d})$ is equivalent to the feasibility of p over $\mathbb{F}^{d \times d}$, uniformly in d and $\mathbb{F} \subseteq \mathbb{C}$.*
- c) *Both f_p and g_p can be computed from p by a Turing machine in time polynomial in the length of p .*

Proof. a) Using addition chains, every integer constant k in p may be replaced by an expression over $(-, \times, 0, 1)$ of length linear in the binary length of k . Thus it is no loss of generality to presume that p is composed from constants 0, 1 and operations $-, \times, \dagger$. These can be expressed by quantum logic formulas according to Fact 48; hence their composition expresses p .

b) Add to f_p conditions expressing Equation (17).

c) The above constructions are easily seen computable in polynomial time. \square

4.5 Satisfiability in Dimensions ≥ 3 is Complete for Real Polynomial Feasibility

In view of Fact 45b), Theorem 44c) means that deciding fixed-dimensional [weak] quantum satisfiability is computationally at most as hard as real polynomial feasibility. Theorem 51 of this section shows that the converse holds as well: Deciding d -dimensional satisfiability of a given quantum formula for $d \geq 3$ is computationally at least as hard as real polynomial feasibility.

A similar result is known for the realizability of a given oriented rank-3 matroid; [BL*99, LEMMA 8.7.1] and, too, based on the above *von Staudt* construction [Rich96, FIGURE 11.7.1].

More generally, consider the following variant of the problem in Equation (13) of Fact 45:

Definition 50. Let $p(Y_1, \dots, Y_n)$ denote a noncommutative $*$ -polynomial with integer coefficients, i.e. an element of the free associative \mathbb{Z} -algebra generated by $Y_1, \dots, Y_n, Y_1^\dagger, \dots, Y_n^\dagger$.

Noncommutative $*$ -Polynomial Feasibility (in dimension $d \in \mathbb{N}$ over field $\mathbb{F} \subseteq \mathbb{C}$) is the computational problem of deciding, given such p (w.l.o.g. of total degree at most 4 in dense encoding[§] and coefficients in $\{0, \pm 1, \pm 2\}$), whether there exist matrices $T_1, \dots, T_n \in \mathbb{F}^{d \times d}$ such that $p(T_1, \dots, T_n) = 0$.

Similarly, **Noncommutative $*$ -Polynomial Identity** is the question of whether to a given such p it holds $p(T_1, \dots, T_n) = 0$ for all $T_1, \dots, T_n \in \mathbb{F}^{d \times d}$.

Note that the famous Amitsur-Levitzki Theorem establishes the noncommutative multilinear polynomial

$$\prod_{\pi \in S_{2n}} \text{sign}(\pi) \cdot \prod_{i=1}^{2n} X_{\pi(i)}$$

to be an identity in $\mathbb{F}^{d \times d}$. And for exploring extensions of this result [Giam90], algorithmically deciding noncommutative $*$ -polynomial identity/feasibility is of practical relevance [BDDK03].

- Theorem 51.** a) Real polynomial feasibility is polynomial-time reducible to noncommutative real $*$ -polynomial feasibility in dimension $d \in \mathbb{N}$.
b) Real polynomial feasibility is also polynomial-time reducible to noncommutative complex $*$ -polynomial feasibility in dimension $d \in \mathbb{N}$.
c) Noncommutative $*$ -polynomial feasibility in dimension d over $\mathbb{F} \subseteq \mathbb{C}$ is polynomial-time reducible to strong satisfiability over $\text{Gr}(\mathbb{F}^{3d})$.
d) Real polynomial feasibility is polynomial-time reducible to strong satisfiability over $\text{Gr}(\mathbb{R}^d)$ and over $\text{Gr}(\mathbb{C}^d)$ for $d \geq 3$.

The above reductions work independent of d and, for Item c), of \mathbb{F} .

Proof. a+b) Consider the given sum of squares $p = \sum_{k=1}^K p_k^2(\vec{X})$ of quadratic multivariate polynomials p_k as a noncommutative polynomial. Note that $p(\vec{X}) = 0$ iff $0 = p_1(\vec{X}) = \dots = p_K(\vec{X})$. Add to p terms $q_\ell \cdot q_\ell^\dagger = 0$ for polynomials $q_\ell := X_\ell - X_\ell^\dagger$ and $q_{\ell,j} = X_\ell \cdot X_j - X_j \cdot X_\ell$: this requires any solution tuple of matrices $X_j \in \mathbb{R}^{d \times d}$ to be symmetric and pairwise commuting. By the Spectral Theorem, any such tuple is simultaneously diagonalizable, i.e. jointly unitarily equivalent to diagonal real (!) matrices Y_j : essentially a d -fold direct product of elements from \mathbb{R} . Each component of this tuple \vec{Y} thus constitutes a root of the 1D case, that is of the original commutative polynomial; and vice versa.

c) see Corollary 49.

d) The case of dimension d a multiple of 3 has been treated in a-c). It thus suffices to replace the quantum logic equation “ $f(\vec{X}) = 1$ ” obtained there by “ $f(\vec{X}) = Z_4$ ” with new variable $Z_4 \in \text{Gr}(\mathbb{F}^d)$ having dimension a positive multiple of 3. The latter condition can be encoded as $Z_0 \vee Z_1 \vee Z_2 \vee Z_3 = 1$ and $Z_j \wedge \bigvee_{i \neq j} Z_i = 0$ for $j = 0, 1, 2, 3$ and $\dim(Z_0) \leq \dim(Z_1) = \dim(Z_2) = \dim(Z_3)$ and $Z_4 = Z_1 \vee Z_2 \vee Z_3$, all understood as abbreviations according to Example 7g). \square

From Theorem 51 and Theorem 43, it follows

Corollary 52. For any fixed $d \geq 3$, [weak] satisfiability over $\text{Gr}(\mathbb{R}^d)$ and over $\text{Gr}(\mathbb{C}^d)$ are mutually polynomial-time equivalent and $\text{BP}(\mathcal{NP}_{\mathbb{R}}^0)$ -complete.

[§] that is an enumeration of all $\mathcal{O}(n^4)$ coefficients to monomials $Z_1 \cdot Z_2 \cdot Z_3 \cdot Z_4$ with $Z_1, \dots, Z_4 \in \{0, 1, Y_1, \dots, Y_n, Y_1^\dagger, \dots, Y_n^\dagger\}$

The proofs of Theorems 51 and 43 also reveal the following consequence which may be of interest of its own to the BSS community:

Scholium 53. *For both $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$ and fixed $d \geq 3$, [weak] satisfiability over $\text{Gr}(\mathbb{F}^d)$ of a given quantum logic formula with constants is $\mathcal{NP}_{\mathbb{R}}$ -complete.*

Again, we stress the similarity to the realizability problem for oriented matroids [BL*99, LEMMA 8.7.1]:

Fact 54. *a) Realizability of a given oriented rank-3 matroid is complete for the class $\text{BP}(\mathcal{NP}_{\mathbb{R}}^0)$.
b) Similarly the problem of deciding, given an oriented rank-3 matroid with some points pre-assigned to coordinates in Euclidean space, whether this assignment can be extended to a realization, is $\mathcal{NP}_{\mathbb{R}}$ -complete.*

4.6 Quantum Logic over Fields other than \mathbb{R} and \mathbb{C}

Definition 1 had to be restricted to the fields of real or complex numbers in order for the scalar product to induce a nonnegative real-valued norm; recall Equation (12). By Theorem 44a), it suffices to focus on the field $\mathbb{A}_{\mathbb{F}}$ of algebraic numbers/algebraic reals.

Orthogonality, however, can be defined for finite-dimensional vector spaces over other fields \mathbb{F} by letting $\neg V = \{\vec{y} : \sum_i x_i y_i = 0 \ \forall \vec{x} \in V\}$; cmp. e.g. [FaFr00]. On the other hand, for fields of positive characteristic, it may well happen that $V \cap \neg V \supsetneq \{0\}$:

Example 55. *For the two-element field $\mathbb{F}_2 = \{0, 1\}$, $\text{Gr}(\mathbb{F}_2^3)$ is basically the Fano Plane: one-dimensional subspaces correspond to the 7 vertices $\{0, 1\}^3 \setminus \{(0, 0, 0)^{\dagger}\}$, two-dimensional subspaces to the 7 edges*

$$\begin{aligned} \left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\} &= \neg \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, & \left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\} &= \neg \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, & \left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\} &= \neg \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \\ \left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\} &= \neg \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, & \left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\} &= \neg \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, & \left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\} &= \neg \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \\ \left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\} &= \neg \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \end{aligned}$$

Note that three vertices are contained in their own orthogonal complement!

Concerning fields of characteristic zero, we have the following counterpart to Theorem 44a):

Corollary 56. *a) To any finite algebraic field extension \mathbb{F} of \mathbb{Q} , there exists a quantum formula f which is [weakly] satisfiable over $\text{Gr}(\mathbb{F}^3)$ but not over $\text{Gr}(\mathbb{Q}^3)$.
b) [Weak] satisfiability over $\text{Gr}(\mathbb{Q}^3)$ is polynomial-time equivalent to the feasibility over \mathbb{Q} of a given multivariate integer polynomial (and hence not known decidable or not: recall Fact 45f).*

Compare the similar effects for Oriented Matroids [BL*99, pp.354-355].

Proof (Corollary 56). Translate a given integer polynomial p into a quantum logic formula g_p according to Corollary 49b): For any field $\mathbb{F} \subseteq \mathbb{C}$, g_p will be satisfiable over $\text{Gr}(\mathbb{F}^3)$ iff p admits a root in \mathbb{F} . Concerning weak satisfiability, observe that the reduction employed in Theorem 43b) based on Lemma 40f) holds uniformly in \mathbb{F} . \square

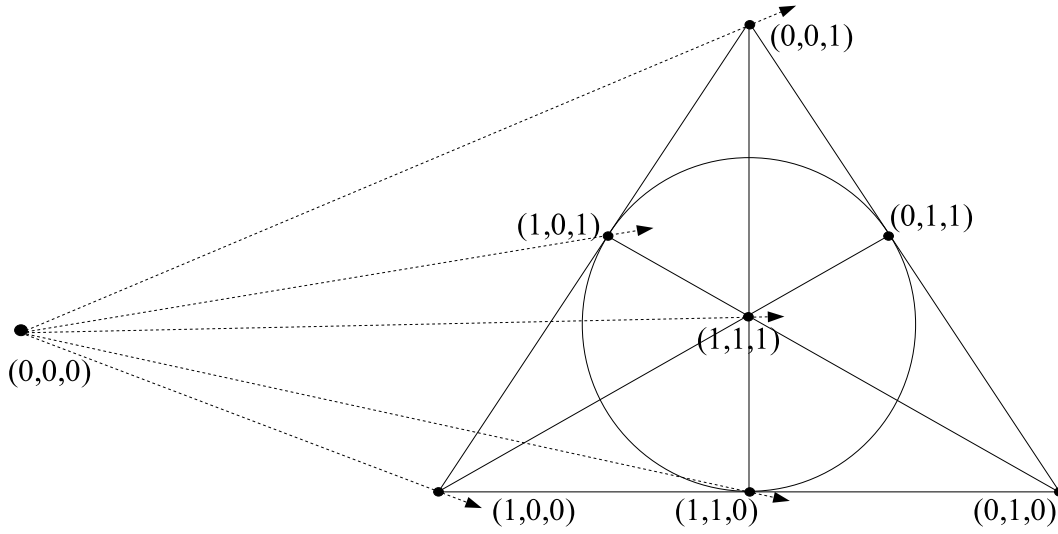


Fig. 5. The Fano Plane

Example 57. In view of Example 7f), abbreviate $J(X, Y, Z) := \neg X \vee (X \wedge (Y \vee Z))$ for incidence and $N(X, Y, Z) := X \vee Y \vee Z$ for non-incidence.

a) The following 9-variate formula is satisfiable over $\text{Gr}(\mathbb{Q}(\sqrt{5})^3)$ but not over $\text{Gr}(\mathbb{Q}^3)$:

$$\begin{aligned} & J(E, A, B) \wedge J(F, A, B) \wedge J(G, A, D) \wedge J(I, A, H) \wedge J(H, B, C) \wedge \\ & \wedge J(I, B, G) \wedge J(G, C, E) \wedge J(H, D, F) \wedge J(I, C, F) \wedge J(I, D, E) \wedge \\ & \wedge N(I, A, B) \wedge N(I, G, H) \wedge (\neg A \vee \neg B) \wedge \bigwedge_{X, Y \in \{A, B, \dots, I\}} (X \vee \neg Y) \end{aligned}$$

b) There exists a 12-variate formula satisfiable over $\text{Gr}(\mathbb{C}^3)$ but not over $\text{Gr}(\mathbb{R}^3)$.

Proof. a) Consider Figure 6 which is well-known to be realizable a projective configuration over coordinate field $\mathbb{Q}(\sqrt{5})$ but not over \mathbb{Q} : cf. e.g. [Grue03, SECTION 5.5.3] or [BL*99, FIGURE 8.4.1].

It remains to observe that any satisfying assignment consists of one-dimensional arguments (i.e. of projective points) only, asserted by the last conjunction.

b) Consider the polynomial equation $X^2 + 1 = 0$ satisfiable over \mathbb{C} but not over \mathbb{R} . Instead of directly turning this into a quantum logic formula by mining the proof of Corollary 49b), we refer to the geometric system of incidences and non-incidences as in Item a). Its infeasibility over \mathbb{R} is best experienced interactively[¶] under

<http://www.mathematik.tu-darmstadt.de/~ziegler/Complex.html>

by trying to move the green point labelled X such as for the gray point labelled $X^2 + 1$ to coincide with 0. \square

We wonder whether, similarly to Proposition 42d), it might happen for a formula to be satisfiable over $\text{Gr}(\mathbb{A}^k)$ as well as over $\text{Gr}(\mathbb{Q}^d)$ for some $d > k$ but not over $\text{Gr}(\mathbb{Q}^k)$. Recall from Equation (12) that, for $\vec{x} \in \mathbb{C}^d$, the norm always satisfies $\langle \vec{x}, \vec{x} \rangle \in \mathbb{R}$; whereas for $\vec{x} \in \mathbb{A}^d$, it usually holds $\langle \vec{x}, \vec{x} \rangle \notin \mathbb{Q}$.

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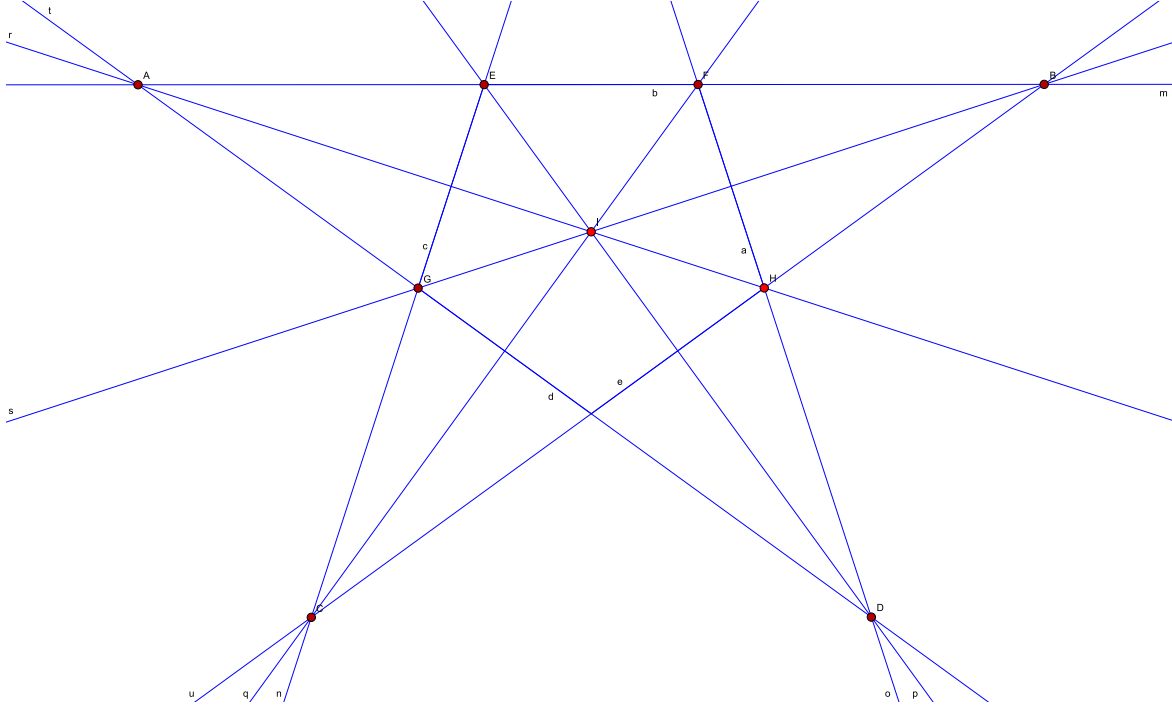


Fig. 6. A configuration in the real projective plane realizable over $\mathbb{Q}(\sqrt{5})$ but not over \mathbb{Q} .

4.7 First-Order Quantum Logic: no Quantifier Elimination but Model-Complete

We now show (Theorem 62c) that quantum logic in three or higher dimensions does not admit quantifier elimination. More precisely consider the algebra $(\text{Gr}(\mathbb{F}^d), \text{Gr}(\mathbb{F}^d), \vee, \wedge, \neg, =)$, where the second entry indicates the availability (in addition to the usual 0 and 1) of all constants. Then for $d \geq 3$ and $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, Theorem 62b+e) shows that this algebra is bi-interpretable to $(\mathbb{R}, \mathbb{R}, +, -, \times, =)$: the *unordered* theory of a real (but not algebraically) closed field with constants. Here, *ordered* inequality “ $a \geq 0$ ” is not expressible; but can be expressed using first-order existential quantification. It follows (Corollary 63) that quantum logic with constants is model-complete.

Definition 58. Let $\mathbb{F} \subseteq \mathbb{C}$ be a field.

- a) For $X_1, \dots, X_n \in \text{Gr}(\mathbb{F}^d)$, write $\vec{i} := \dim(\vec{X}) := (\dim(X_1), \dots, \dim(X_n))$. We say that \vec{X} is a \vec{i} -dimensional tuple of subspaces of \mathbb{F}^d and write $\text{Gr}_{\vec{i}}(\mathbb{F}^d) := \prod_{j=1}^n \text{Gr}_{i_j}(\mathbb{F}^d)$.
- b) For $g(\vec{X}) = f(c_1, \dots, c_m; X_1, \dots, X_n)$ a formula with constants $c_1, \dots, c_m \in \text{Gr}(\mathbb{F}^d) \ni Z$ and $\vec{i} = (i_1, \dots, i_n)$ with $i_j \in \{0, 1, \dots, d\}$, consider the \vec{i} -dimensional **preimage**

$$g_{\vec{i}}^{-1}[\geq Z] := \{(X_1, \dots, X_n) : X_j \in \text{Gr}_{i_j}(\mathbb{F}^d), g(\vec{X}) \geq Z\} \subseteq \text{Gr}_{\vec{i}}(\mathbb{F}^d).$$

Similarly for $g_{\vec{i}}^{-1}[\leq Z]$ and $g_{\vec{i}}^{-1}[=Z]$.

- c) A set $S \subseteq \mathbb{F}^d$ is **constructible** over $\mathbb{E} \subseteq \mathbb{F}$ if there exist finitely many polynomials $p_1, \dots, p_N \in \mathbb{E}[Y_1, \dots, Y_d]$ such that S is a Boolean combination (set-theoretic union, intersection, and complement) of the sets $\{(y_1, \dots, y_d) : y_i \in \mathbb{F}, p_j(\vec{y}) = 0\}$.
In case $\mathbb{F} \subseteq \mathbb{R}$, S is **semi-algebraic** over \mathbb{E} if it is a Boolean combination of finitely many sets $\{(y_1, \dots, y_d) : y_i \in \mathbb{F}, p_j(\vec{y}) \geq 0\}$, $p_j \in \mathbb{E}[Y_1, \dots, Y_d]$.

- d) Let $\mathbb{PF}^d := \text{Gr}_1(\mathbb{F}^d)$ denote $(d-1)$ -dimensional **projective space**. A set $S \subseteq \mathbb{PF}^d$ is **constructible (over \mathbb{E})** if it is a Boolean combination of finitely many sets $\{(y_1, \dots, y_d) : y_i \in \mathbb{F}, p_j(\vec{y}) = 0\}$ where p_j now are homogeneous polynomials (over \mathbb{E}). Similarly in case $\mathbb{F} \subseteq \mathbb{R}$, S is **semi-algebraic (over \mathbb{E})** if it is a Boolean combination of finitely many sets $\{(y_1, \dots, y_d) : y_i \in \mathbb{F}, p_j(\vec{y}) \geq 0\}$ with p_j homogeneous polynomials (over \mathbb{E}) of even degree.
- e) Call a polynomial $p(\vec{x}_1, \dots, \vec{x}_k)$ in variable blocks $\vec{x}_i = (x_{i,1}, \dots, x_{i,d_k})$ **block-homogeneous** of degree $(\delta_1, \dots, \delta_k) \in \mathbb{N}^k$ if $p(\lambda_1 \vec{x}_1, \dots, \lambda_k \vec{x}_k) = \lambda_1^{\delta_1} \dots \lambda_k^{\delta_k} \cdot p(\vec{x}_1, \dots, \vec{x}_k)$. $S \subseteq \mathbb{PF}^{d_1} \times \dots \times \mathbb{PF}^{d_k}$ is **constructible (over \mathbb{E})** if it is a Boolean combination of finitely many sets $\{(\vec{y}_1, \dots, \vec{y}_k) : \vec{y}_i \in \mathbb{PF}^{d_i}, p_j(\vec{y}_1, \dots, \vec{y}_k) = 0\}$ with p_j block-homogeneous (over \mathbb{E}). S is **semialgebraic (over $\mathbb{E} \subseteq \mathbb{F} \cap \mathbb{R}$)** if it is a Boolean combination of finitely many sets $\{(\vec{y}_1, \dots, \vec{y}_k) : \vec{y}_i \in \mathbb{PF}^{d_i}, p_j(\vec{y}_1, \dots, \vec{y}_k) \geq 0\}$ with p_j block-homogeneous (over \mathbb{E}) of componentwise even degree. Here “ $x + iy \geq 0$ ” is to be understood as “ $x \geq 0$ and $y = 0$ ”.
- f) $S \subseteq \mathbb{F}^d$ is ***-constructible** over \mathbb{E} if it is a Boolean combination of finitely many $\{\vec{z} : p_j(z_1, \dots, z_d, z_1^*, \dots, z_d^*) = 0\}$, where $(x + iy)^* = x - iy$ denotes complex conjugation. S is ***-semialgebraic** (over $\mathbb{E} \subseteq \mathbb{F} \cap \mathbb{R}$) if it is a Boolean combination of finitely many sets $\{\vec{z} : p_j(z_1, \dots, z_d, z_1^*, \dots, z_d^*) \geq 0\}$. Similarly for *-constructible and *-semialgebraic subsets of projective space.
- g) For $\vec{x}_1, \dots, \vec{x}_k \in \mathbb{F}^d$ linearly independent, the **Plücker Coordinates** of $L := \text{lspan}(\vec{x}_1, \dots, \vec{x}_k) \in \text{Gr}_k(\mathbb{F}^d)$ are the homogeneous vector

$$\tilde{L} := \det(\vec{x}_{j_1}, \dots, \vec{x}_{j_d})_{1 \leq j_1 < \dots < j_d \leq n} \in \mathbb{PF}^{d(d-k)}. \quad (18)$$

For $\vec{i} = (i_1, \dots, i_n)$ and $\mathcal{S} \subseteq \text{Gr}_i(\mathbb{F}^d)$, write $\tilde{\mathcal{S}} := \{(\tilde{L}_1, \dots, \tilde{L}_n) : (L_1, \dots, L_n) \in \mathcal{S}\} \subseteq \mathbb{PF}^{d(d-i_1)} \times \dots \times \mathbb{PF}^{d(d-i_n)}$.

Definition 58c-e+g) is common in algebraic geometry [KILa72, BPR03]. Being constructible (sometimes also called quasi-algebraic/quasi-projective) amounts to a finite union of sets locally closed in the Zariski topology.

Fact 59. a) A constructible subset of \mathbb{F} or \mathbb{PF}^2 is either finite or cofinite.

b) If $S \subseteq \mathbb{R}^d$ is decidable in constant time (i.e. in a number of steps possibly depending on the length d , but independent of the value of the input $x \in \mathbb{R}^d$) by a BSS-machine [devoid of constants], then S is constructible [over $\mathbb{E} := \{-1, 0, +1\}$].

c) For $B \subseteq \mathbb{R}^m \times \mathbb{R}^n$ semi-algebraic over \mathbb{E} , the projection

$$\pi_m B := \{\vec{y} \in \mathbb{R}^n : \exists \vec{x} \in \mathbb{R}^m : (\vec{x}, \vec{y}) \in B\} \quad (19)$$

is again semi-algebraic over \mathbb{E} : recall the **Tarski-Seidenberg Principle** as, e.g., in [BPR03, THEOREM 2.80].

- d) **Plücker Coordinates** of $\text{lspan}(\vec{x}_1, \dots, \vec{x}_k)$ are projectively well-defined: To $\vec{y}_1, \dots, \vec{y}_k \in \mathbb{F}^d$ with $\text{lspan}(\vec{y}_1, \dots, \vec{y}_k) = \text{lspan}(\vec{x}_1, \dots, \vec{x}_k)$, there exists $c \in \mathbb{F} \setminus \{0\}$ with $\det(\vec{x}_{j_1}, \dots, \vec{x}_{j_d}) = c \cdot \det(\vec{y}_{j_1}, \dots, \vec{y}_{j_d})$ for each $1 \leq j_1 < \dots < j_d \leq n$; cf. e.g., [KILa72, p.1063].
- e) **Plücker Coordinates** are unique: For $X, Y \in \text{Gr}_k(\mathbb{F}^d)$, $\tilde{X} = \tilde{Y}$ implies $X = Y$. From $\tilde{X} = (x_{j_1, \dots, j_k})_{1 \leq j_1 < \dots < j_k \leq d} \in \mathbb{PF}^{d(d-k)}$ with $X \in \text{Gr}_k(\mathbb{F}^d)$, one can recover $X = \text{range}(A)$ with $A \in \mathbb{F}^{d \times k}$ defined by $a_{rs} := x_{\text{sort}(i_1, \dots, i_{s-1}, r, i_{s+1}, \dots, i_k)} \cdot \text{sign}(i_1, \dots, i_{s-1}, r, i_{s+1}, \dots, i_k)$, where $1 \leq i_1 < \dots < i_k \leq d$ are fixed with $x_{i_1, \dots, i_k} \neq 0$ and $\text{sort}(j_1, \dots, j_k)$ denotes

(j_1, \dots, j_k) sorted ascendingly; $\text{sign}(j_1, \dots, j_k) = \pm 1$ the sign of the unique permutation yielding this sorting in case $\text{Card}\{j_1, \dots, j_k\} = k$, $\text{sign}(j_1, \dots, j_k) := 0$ otherwise. [KILa72, p.1065]

f) For $X \in \text{Gr}_1(\mathbb{F}^d)$, it holds $\tilde{X} = X$. In particular to $\bar{x} \in \mathbb{P}\mathbb{F}^d$, there exist $X \in \text{Gr}_1(\mathbb{F}^d)$ and $Y \in \text{Gr}_{d-1}(\mathbb{F}^d)$ with $\bar{x} = \tilde{X} = \tilde{Y}$. However for $1 < k < d$, $\widehat{\text{Gr}_k(\mathbb{F}^d)}$ is a proper subset of $\mathbb{P}\mathbb{F}^{d(d-k)}$, defined by the so-called *Plücker Relations*. These reveal that $\widehat{\text{Gr}_k(\mathbb{F}^d)}$ is constructible over $\{-1, 0, +1\}$ [KILa72, p.1065].

Concerning a), recall that a nonzero polynomial has in the field \mathbb{F} only finitely many roots; and co-/finiteness is invariant under complement, binary union, and binary intersection. Claim b) follows from a standard path decomposition; cmp. [BCSS98, THEOREM 2.3.1(1)] (division $Y = 1/Y$ for instance may be expressed as multiplication $X \cdot Y = 1$) and remember that we are in the unordered setting. Surprisingly, it suffices that S be merely BSS-decidable: cf. [CuRo93].

Example 60. a) The set $\{(\vec{x}_1, \dots, \vec{x}_k) : \langle \vec{x}_j, \vec{x}_\ell \rangle = \delta_{j,\ell}\} \subseteq \mathbb{R}^{k \times d}$ of k -tuples of orthonormal vectors is constructible over $\{-1, 0, +1\}$.

b) Let $S_j \subseteq \mathbb{F}^{d_j}$ ($j = 1, \dots, k$). Then $\prod_{j=1}^k S_j \subseteq \mathbb{R}^{\sum_j d_j}$ is constructible iff each S_j is.

c) For $\mathbb{F} \subseteq \mathbb{R}$, a set $S \subseteq (\mathbb{F} + i\mathbb{F})^d$ is $*$ -constructible [$*$ -semialgebraic] over $\mathbb{E} \subseteq \mathbb{F}$ iff $\hat{S} := \{(\text{Re } z_1, \text{Im } z_1, \dots, \text{Re } z_d, \text{Im } z_d) : (z_1, \dots, z_d) \in S\} \subseteq \mathbb{F}^{2d}$ is constructible [semi-algebraic] over \mathbb{E} . Similarly for subsets of $\mathbb{P}(\mathbb{F} + i\mathbb{F})^d$. In particular, constructible/semialgebraic is equivalent to $*$ -constructible/ $*$ -semialgebraic in the real case.

d) In view of Fact 59a), the set $[0, \infty) \subseteq \mathbb{R}$ cannot be constructible. It is, however, semi-algebraic.

e) In fact, each set $A \subseteq \mathbb{R}^n$ semi-algebraic over \mathbb{E} is the projection $A = \pi_m B$ of a set $B \subseteq \mathbb{R}^m \times \mathbb{R}^n$ constructible over \mathbb{E} : “ $p(\vec{x}) \geq 0$ ” can be rewritten as “ $\exists y : p(\vec{x}) - y^2 = 0$ ”.

f) Similarly, to $A \subseteq \mathbb{P}\mathbb{F}^{d_1} \times \dots \times \mathbb{P}\mathbb{F}^{d_k}$ $*$ -semialgebraic over $\mathbb{E} \subseteq \mathbb{F} \cap \mathbb{R}$, there exists $d_0 \in \mathbb{N}$ and $B \subseteq \mathbb{P}\mathbb{F}^{d_0} \times \mathbb{P}\mathbb{F}^{d_1} \times \dots \times \mathbb{P}\mathbb{F}^{d_k}$ $*$ -constructible over \mathbb{E} such that A is the projection $\pi_{d_0} B$ of B , where

$$\pi_{d_0} B := \{(\bar{x}_1, \dots, \bar{x}_k) \in \mathbb{P}\mathbb{F}^{d_1} \times \dots \times \mathbb{P}\mathbb{F}^{d_k} : \exists \bar{x}_0 \in \mathbb{P}\mathbb{F}^{d_0} : (\bar{x}_0, \bar{x}_1, \dots, \bar{x}_k) \in B\} \quad (20)$$

g) Based on Fact 59e), it is easy to program a BSS machine devoid of constants which, given (real and imaginary parts of) $\tilde{X} \in \mathbb{F}^{d(d-k)} \setminus \{0\}$, calculates a matrix $A \in \mathbb{F}^{d \times d}$ with $\text{range}(A) = X \in \text{Gr}_k(\mathbb{F}^d)$.

We now relate constructible/semialgebraic subsets of projective space to similar subsets of affine space; and sets of subspaces^{||} to their corresponding projective sets of Plücker Coordinates:

Lemma 61. Fix a field $\mathbb{F} \subseteq \mathbb{C}$. For $\bar{x} = (x_1, \dots, x_d) \in \mathbb{F}\mathbb{P}^d$ with $x_j \neq 0$, abbreviate $\bar{x}/x_j := (x_1/x_j, \dots, 1, \dots, x_d/x_j) \in \mathbb{F}^d$. Moreover write, for $S \subseteq \mathbb{P}\mathbb{F}^d$ and $j \in \{1, \dots, d\}$, $S_j := \{\bar{x}/x_j : \bar{x} \in S, x_j \neq 0\} \subseteq \mathbb{F}^d$. More generally consider $S \subseteq \mathbb{P}\mathbb{F}^{d_1} \times \dots \times \mathbb{P}\mathbb{F}^{d_k}$ and $j_i \in \{1, \dots, d_i\}$, $i = 1, \dots, k$; then let

$$S_{(j_1, \dots, j_k)} := \left\{ \left(\bar{x}_1/x_{1,j_1}, \dots, \bar{x}_k/x_{k,j_k} \right) : (\bar{x}_1, \dots, \bar{x}_k) \in S, x_{i,j_i} \neq 0 \forall i \right\}.$$

^{||} and hope the reader does not get confused by a subspace of \mathbb{F}^d being a set itself

- a) $S \subseteq \mathbb{P}\mathbb{F}^{d_1} \times \dots \times \mathbb{P}\mathbb{F}^{d_k}$ is constructible/*-constructible/semialgebraic/*-semialgebraic over \mathbb{E} iff, for each $\vec{j} = (j_1, \dots, j_k)$ with $j_i \in \{1, \dots, d_i\}$, $S_{\vec{j}} \subseteq \mathbb{F}^{d_1} \times \dots \times \mathbb{F}^{d_k}$ is.
- b) Let $T \subseteq \mathbb{P}\mathbb{F}^{d_0} \times \mathbb{P}\mathbb{F}^{d_1} \times \dots \times \mathbb{P}\mathbb{F}^{d_k}$ and $j_i \in \{1, \dots, d_i\}$ for $i = 1, \dots, k$. Then

$$(\pi_{d_0} T)_{(j_1, \dots, j_k)} = \bigcup_{j_0=1}^{d_0} \pi_{d_0}(T_{(j_0, j_1, \dots, j_k)}) .$$

- c) If $T \subseteq \mathbb{P}\mathbb{F}^{d_0} \times \mathbb{P}\mathbb{F}^{d_1} \times \dots \times \mathbb{P}\mathbb{F}^{d_k}$ is *-semialgebraic over \mathbb{E} , then so is $\pi_{d_0} T \subseteq \mathbb{P}\mathbb{F}^{d_1} \times \dots \times \mathbb{P}\mathbb{F}^{d_k}$.
- d) For $\vec{i} \in \{0, 1, \dots, d\}^n$ and $\mathcal{S}, \mathcal{T} \subseteq \text{Gr}_{\vec{i}}(\mathbb{F}^d)$, $\tilde{\mathcal{S}} = \tilde{\mathcal{T}}$ implies $\mathcal{S} = \mathcal{T}$. Moreover it holds

$$\tilde{\mathcal{S}} \cup \tilde{\mathcal{T}} = \widetilde{\mathcal{S} \cup \mathcal{T}}, \quad \tilde{\mathcal{S}} \cap \tilde{\mathcal{T}} = \widetilde{\mathcal{S} \cap \mathcal{T}}, \quad \widetilde{\text{Gr}_{\vec{i}}(\mathbb{F}^d) \setminus \tilde{\mathcal{S}}} = \widetilde{\text{Gr}_{\vec{i}}(\mathbb{F}^d) \setminus \mathcal{S}} .$$

If $\tilde{\mathcal{S}}, \tilde{\mathcal{T}}$ are *-constructible/*-semialgebraic, then so are $\widetilde{\mathcal{S} \cup \mathcal{T}}$, $\widetilde{\mathcal{S} \cap \mathcal{T}}$, and $\widetilde{\text{Gr}_{\vec{i}}(\mathbb{F}^d) \setminus \tilde{\mathcal{S}}}$.

- e) Let $0 \leq i, j \leq d$ and $\mathcal{T} \subseteq \text{Gr}_i(\mathbb{F}^d) \times \text{Gr}_j(\mathbb{F}^d) = \text{Gr}_{i,j}(\mathbb{F}^d)$. Then the projection $\pi_i \mathcal{T} := \{Y \in \text{Gr}_j(\mathbb{F}^d) : \exists X \in \text{Gr}_i(\mathbb{F}^d) : (X, Y) \in \mathcal{T}\}$ has Plücker coordinate set $\widetilde{\pi_i \mathcal{T}} = \pi_i \tilde{\mathcal{T}}$; recall Equation (20).

More generally let $\vec{i} = (i_1, \dots, i_n)$ and $\vec{j} = (j_1, \dots, j_m)$, $\mathcal{T} \subseteq \text{Gr}_{\vec{i}, \vec{j}}(\mathbb{F}^d)$. Then the projection

$$\pi_{\vec{i}} \mathcal{T} := \{\vec{Y} \in \text{Gr}_{\vec{j}}(\mathbb{F}^d) : \exists \vec{X} \in \text{Gr}_{\vec{i}}(\mathbb{F}^d) : (\vec{X}, \vec{Y}) \in \mathcal{T}\} \quad (21)$$

has $\widetilde{\pi_{\vec{i}} \mathcal{T}} = \pi_{i_1} \pi_{i_2} \dots \pi_{i_n} \tilde{\mathcal{T}}$.

- f) Let $\mathcal{T} \subseteq \text{Gr}_{\vec{i}, \vec{j}}(\mathbb{F}^d)$ be such that $\tilde{\mathcal{T}}$ is *-semialgebraic over \mathbb{E} . Then $\mathcal{S} := \pi_{\vec{i}} \mathcal{T} \subseteq \text{Gr}_{\vec{j}}(\mathbb{F}^d)$ has $\tilde{\mathcal{S}}$ again *-semialgebraic over \mathbb{E} .

Conversely to $\mathcal{S} \subseteq \text{Gr}_{\vec{j}}(\mathbb{F}^d)$ with $\tilde{\mathcal{S}}$ *-semialgebraic over \mathbb{E} , there exists $n \in \mathbb{N}$ and $\mathcal{T} \subseteq \text{Gr}_{\vec{i}, \vec{j}}(\mathbb{F}^d)$ with $\tilde{\mathcal{T}}$ *-constructible over \mathbb{E} such that $\mathcal{S} = \pi_{\vec{i}} \mathcal{T}$ with the n -tuple $\vec{i} := (1, \dots, 1)$.

- g) For $Y, Z \in \text{Gr}(\mathbb{F}^d)$, $\vec{i} = (i_1, \dots, i_n) \in \{0, 1, \dots, d\}^n$, and n -variate formulas f, g , it holds

- i) $f_{\vec{i}}^{-1}[\leq Z] = (\neg f)_{\vec{i}}^{-1}[\geq \neg Z]$ and $f^{-1}[\leq Z] = f^{-1}[\geq Z] \cap f^{-1}[\leq Z]$.
- ii) $f_{\vec{i}}^{-1}[\leq Y \wedge Z] = f_{\vec{i}}^{-1}[\leq Y] \cap f_{\vec{i}}^{-1}[\leq Z]$ and $f_{\vec{i}}^{-1}[\geq Y \vee Z] = f_{\vec{i}}^{-1}[\geq Y] \cap f_{\vec{i}}^{-1}[\geq Z]$.
- iii) $(f \wedge g)_{\vec{i}}^{-1}[\geq Z] = f_{\vec{i}}^{-1}[\geq Z] \cap g_{\vec{i}}^{-1}[\geq Z]$ and $(f \vee g)_{\vec{i}}^{-1}[\leq Z] = f_{\vec{i}}^{-1}[\leq Z] \cap g_{\vec{i}}^{-1}[\leq Z]$.

Proof. a) In view of Example 60c) and since both notions are closed under finite intersection, finite union, and complement, it suffices to consider ‘basic’ sets like, e.g. $S = \{(\bar{x}_1, \dots, \bar{x}_k) : p(\bar{x}_1, \dots, \bar{x}_k) = 0\}$ with p block-homogeneous of degree $(\delta_1, \dots, \delta_k)$. In fact let us start with the case $k = 1$ and note that, for $x_j \neq 0$, $p(x_1, \dots, x_d) = x_j^\delta \cdot p(x_1/x_j, \dots, x_d/x_j)$ is zero iff $p(x_1/x_j, \dots, x_d/x_j)$ is. Similarly it holds $p(x_1, \dots, x_d) \geq 0 \Leftrightarrow p(x_1/x_j, \dots, x_d/x_j) \geq 0$, because $x_j^\delta \geq 0$ (δ even, recall Definition 58). More generally, $p(\bar{x}_1, \dots, \bar{x}_k) = x_{j_1}^{\delta_1} \dots x_{j_k}^{\delta_k} \cdot p(\bar{x}_1/x_{j_1}, \dots, \bar{x}_k/x_{j_k})$ shows that S constructible/semialgebraic over \mathbb{E} has $S_{\vec{j}}$ constructible/semialgebraic over \mathbb{E} , again: because $\{(\vec{x}_1, \dots, \vec{x}_k) : x_{i,j_i} \neq 0 \forall i\} \subseteq \mathbb{F}^{d_1} \times \dots \times \mathbb{F}^{d_k}$ is obviously constructible.

Conversely suppose that $S_j \subseteq \mathbb{F}^d$ is constructible for each $j = 1, \dots, d$. Again w.l.o.g. consider $S_j = \{\vec{x} : p_j(\vec{x}) = 0\}$. Let δ be an upper bound to the maximum degree of p_j . Then $q_j(x_1, \dots, x_d) := p_j(x_1/x_j, \dots, x_d/x_j) \cdot x_j^\delta$ is a polynomial homogeneous of degree δ with $q_j(\vec{x}) = 0 \Leftrightarrow p_j(\vec{x}/x_j) = 0$: provided that $x_j \neq 0$. In case $x_j = 0$, we can conclude only one direction of the equivalence: $q_j(\vec{x}) = 0 \Leftarrow p_j(\vec{x}/x_j) = 0$. However, to $\bar{x} \in \mathbb{P}\mathbb{F}^d$ there exists j with $x_j \neq 0$; hence $S = \bigcap_j \{\bar{x} : q_j(\bar{x}) = 0\}$ is constructible. In the semialgebraic case $S_j = \{\vec{x} : p_j(\vec{x}) \geq 0\}$, choosing δ even also yields $q_j(\bar{x}) \geq 0 \Leftrightarrow p_j(\bar{x}/x_j) \geq 0$; and the provision $x_j \neq 0$ can be removed similarly as before and reveals that S is semialgebraic.

b) Note that

$$\begin{aligned} & \left\{ (\bar{x}_1, \dots, \bar{x}_k) \in \mathbb{P}\mathbb{F}^{d_1} \times \dots \times \mathbb{P}\mathbb{F}^{d_k} : \exists \bar{x}_0 \in \mathbb{P}\mathbb{F}^{d_0} : (\bar{x}_0, \bar{x}_1, \dots, \bar{x}_k) \in T \right\}_{(j_1, \dots, j_k)} = \\ & = \bigcup_{j_0=1}^{d_0} \left\{ (\vec{x}_1, \dots, \vec{x}_k) \in \mathbb{F}^{d_1} \times \dots \times \mathbb{F}^{d_k} : \exists \vec{x}_0 \in \mathbb{F}^{d_0} : (\vec{x}_0, \vec{x}_1, \dots, \vec{x}_k) \in T_{(j_0, j_1, \dots, j_k)} \right\} . \end{aligned}$$

Indeed, $(\bar{x}_1/x_{1,j_1}, \dots, \bar{x}_k/x_{k,j_k})$ belongs to either side iff $x_{i,j_i} \neq 0$ ($i = 1, \dots, k$) and there exists $\bar{x}_0 \in \mathbb{F}^{d_0}$ and j_0 with $x_{0,j_0} \neq 0$ such that $(\bar{x}_0/x_{0,j_0}, \bar{x}_1/x_{1,j_1}, \dots, \bar{x}_k/x_{k,j_k}) \in T_{(j_0, j_1, \dots, j_k)}$.

- c) In view of a) and Example 60c), it suffices to show that $(\pi_{d_0} T)_{\bar{j}} \subseteq \mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_k}$ is semialgebraic over \mathbb{E} . But this follows from b) and Fact 59c), regarding that semialgebraic sets are closed under finite union.
- d) follows from Fact 59e+f)
- e) follows from Fact 59d+e). Note that the projection of a set of Plücker coordinates is indeed a set of Plücker coordinates again.
- f) For the first claim, combine c) with e). For the converse, recall Example 60e+f) and employ Fact 59e) to write the (say, n) additional coordinates as n 1D subspaces, i.e. embed them into $\text{Gr}_{\bar{i}}(\mathbb{F}^d)$.
- g) immediate. □

Theorem 62. *Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.*

- a) *For n -variate formula f and $\vec{i} \in \{0, 1, \dots, d\}^n$, both sets $f_{\vec{i}}^{-1}[=1]$ and $f_{\vec{i}}^{-1}[\neq 0]$ of [weakly] satisfying assignments in $\text{Gr}_{\vec{i}}(\mathbb{F}^d)$ have sets of Plücker Coordinates $\widetilde{f_{\vec{i}}^{-1}[=1]}, \widetilde{f_{\vec{i}}^{-1}[\neq 0]}$ $\subseteq \mathbb{P}\mathbb{F}^{d(d-i_1)} \times \dots \times \mathbb{P}\mathbb{F}^{d(d-i_n)}$ *-constructible over $\{-1, 0, +1\}$.*
- b) *If f is a formula with constants, then $\widetilde{f_{\vec{i}}^{-1}[=1]}$ and $\widetilde{f_{\vec{i}}^{-1}[\neq 0]}$ are *-constructible over \mathbb{R} .*
- c) *Consider the set $\mathcal{S} := \{(0, x, -ax) : x \in \mathbb{F}\} : \mathbb{R} \ni a \geq 0\} \subseteq \text{Gr}_1(\mathbb{F}^3)$ of 1D subspaces of \mathbb{F}^3 . There exists a bivariate formula $g(X, Y) = g(\vec{c}; X, Y)$ with constants $\vec{c} \in \text{Gr}(\mathbb{F}^3)$ such that it holds*

$$\forall X \in \text{Gr}(\mathbb{F}^3) : \quad (X \in \mathcal{S} \iff \exists Y \in \text{Gr}(\mathbb{F}^3) : g(X, Y) = 1) .$$

However there is no univariate formula f with constants such that $\mathcal{S} = f_1^{-1}[=1]$.

- d) *As a first converse to b), let $d \geq 3$ and $\vec{i} = (1, \dots, 1)$ with $n = |\vec{i}|$ and $\mathcal{S} \subseteq \text{Gr}_{\vec{i}}(\mathbb{F}^d)$ with $\tilde{\mathcal{S}}$ *-constructible over \mathbb{R} . Then there exists an n -variate formula f with constants from $\text{Gr}(\mathbb{F}^d)$ such that $\mathcal{S} = \widetilde{f_{\vec{i}}^{-1}[=1]} = \widetilde{f_{\vec{i}}^{-1}[\neq 0]}$.*
- e) *This time let $\vec{i} \in \{0, \dots, d\}^n$ ($d \geq 3$) and $\mathcal{S} \subseteq \text{Gr}_{\vec{i}}(\mathbb{F}^d)$ with $\tilde{\mathcal{S}}$ *-constructible over \mathbb{R} . Then there exists a formula f with constants from $\text{Gr}(\mathbb{F}^d)$ such that $\mathcal{S} = f_{\vec{i}}^{-1}[=1] = f_{\vec{i}}^{-1}[\neq 0]$.*
- f) *Let $f(\vec{X}, \vec{Y})$ be a $(n+m)$ -variate formula with constants from $\text{Gr}(\mathbb{F}^d)$ and $\vec{j} = (j_1, \dots, j_m)$. Then $\mathcal{S} := \{\vec{Y} \in \text{Gr}_{\vec{j}}(\mathbb{F}^d) : \exists \vec{X} \in \text{Gr}(\mathbb{F}^d) : f(\vec{X}, \vec{Y}) = 1\}$ has $\tilde{\mathcal{S}}$ *-semialgebraic over \mathbb{R} .*
- g) *Conversely, let $d \geq 3$ and $\mathcal{S} \subseteq \text{Gr}_{\vec{j}}(\mathbb{F}^d)$ such that $\tilde{\mathcal{S}}$ is *-semialgebraic over \mathbb{R} . Then there exists a formula $f(\vec{X}, \vec{Y})$ with constants such that $\mathcal{S} = \{\vec{Y} \in \text{Gr}_{\vec{j}}(\mathbb{F}^d) : \exists \vec{X} \in \text{Gr}(\mathbb{F}^d) : f(\vec{X}, \vec{Y}) = 1\}$.*
- h) *To each $d \in \mathbb{N}$ and to each formula $f(\vec{X}, \vec{Y}, \vec{Z})$ with constants from $\text{Gr}(\mathbb{F}^d)$, there exists a formula $g(\vec{X}, \vec{W})$ with constants from $\text{Gr}(\mathbb{F}^d)$ such that, for each $\vec{X} \in \text{Gr}(\mathbb{F}^d)$, it holds:*

$$\forall \vec{Y} \in \text{Gr}(\mathbb{F}^d) \exists \vec{Z} \in \text{Gr}(\mathbb{F}^d) : f(\vec{X}, \vec{Y}, \vec{Z}) = 1 \iff \exists \vec{W} \in \text{Gr}(\mathbb{F}^d) : g(\vec{X}, \vec{W}) = 1 .$$

Iterated application of h) yields model-completeness of quantum logic:

Corollary 63. *In fixed dimensions, any quantified formula (possibly with constants) is weakly equivalent to some purely existentially quantified one with constants.*

Question 64. *Is fixed-dimensional quantum logic without constants model complete as well?*

Note that for $\mathcal{S} \subseteq \text{Gr}_i(\mathbb{F}^d)$ with $\tilde{\mathcal{S}}$ $*$ -constructible over \mathbb{Q} to be of the form $\mathcal{S} = f_i^{-1}[=1]$ or $\mathcal{S} = f_i^{-1}[\neq 0]$, one must at least require \mathcal{S} to be orthogonally invariant: Lemma 40a). Moreover observe that the embedding of ring operations into $\text{Gr}(\mathbb{F}^3)$ is unique not only just up to orthogonal invariance, but also up to ‘scaling’, i.e. up to $X_j(\text{id})$: Example 47d+j).

Proof (Theorem 62).

- a) Combine Example 60g) with Observation 28 to obtain a BSS machine devoid of constants which, given (real and imaginary parts of) nonzero $\vec{X}_1 \in \mathbb{F}^{d(d-i_1)}, \dots, \vec{X}_n \in \mathbb{F}^{d(d-i_n)}$, accepts iff $f(X_1, \dots, X_n) = 1 [\neq 0]$. Hence Fact 59b) and Lemma 61c) show that the sets $\widehat{(f_i^{-1}[\neq 0])}_j$ and $\widehat{(f_i^{-1}[=1])}_j$ are $*$ -constructible over $\{-1, 0, +1\}$.
- b) Similarly to a), but now both f and the BSS machine evaluating it may employ constants from \mathbb{R} .
- c) Note that $\tilde{\mathcal{S}} = \mathcal{S}$ (Fact 59f). In case $\mathbb{F} = \mathbb{R}$, intersection with constructible $\{0\} \times \{1\} \times \mathbb{R}$ shows that constructibility of $\mathcal{S} \subseteq \mathbb{R}^3$ would contradict Example 60d). In case $\mathbb{F} = \mathbb{C}$, similarly intersect the real part to see that \mathcal{S} cannot be $*$ -constructible. Hence a+b) prohibit \mathcal{S} to be of the form $f_1^{-1}[=1]$ or $f_1^{-1}[\neq 0]$.
On the other hand, note that $a \geq 0$ is equivalent “ $\exists b : (b + b^*)^2 - a = 0$ ”. Now Example 47 and Corollary 49a) show how to express this condition over \mathbb{F} by means of a quantum logic formula over $\text{Gr}(\mathbb{F}^3)$ “ $\exists X(b) \in \text{Gr}_1(\mathbb{F}^3) : f(X(b), X(a); W_0, W_1, W_2, V_0, V_1) = W_0$ ” with constants W_0, W_1, W_2, V_0, V_1 .
- d) Recall (Example 47 and Corollary 49) that any polynomial $p \in \mathbb{R}[T_1, \dots, T_m]$ can be encoded as a formula $f_p(X_1, \dots, X_m)$ with constants from $\text{Gr}(\mathbb{F}^d)$ such that it holds for all $T_1, \dots, T_m \in \mathbb{F}$:

$$p(\vec{T}) = 0 \quad \Leftrightarrow \quad f_p(X(T_1), \dots, X(T_m)) = 1 \quad \Leftrightarrow \quad f_p(X(T_1), \dots, X(T_m)) \neq 0 \quad . \quad (22)$$

First consider the case $d = 3$. So we are given n lines $L_k = \{(a_k t, b_k t, c_k t) : t \in \mathbb{F}\}$ with $(a_k, b_k, c_k) \in \mathbb{P}\mathbb{F}^3$. Using formulas with constants $Y_j(L_k) := \Pi(L_k, X_j(0) \vee X^j(0))$, $j = 0, 1, 2$, one obtains the projections of L_k to the principal XY , YZ , and ZX -plane: i.e. subspaces $X_0(-c_k/b_k)$, $X_1(-a_k/c_k)$, and $X_2(-b_k/a_k)$ in the notation of Example 47c); recall also Figure 4. Separately ‘handling’ (by means of the Boolean connectives according to Example 20 and Theorem 35c) degenerate cases like $Y_j(L_k) = 0$ (i.e. $b_k = 0$), one thus essentially obtains the homogeneous ‘coordinate’ vector (a_k, b_k, c_k) of L_k . By hypothesis, their joint membership to \mathcal{S} is defined by a Boolean combination of homogeneous polynomial equalities $p_\ell(a_1, \dots, c_n) = 0$; and these can be rephrased as quantum logic conditions according to Equation (22).

In the higher dimensions, similar projections onto $d+1$ appropriate two-dimensional principal planes yield the homogeneous coordinates of each lines L .

- e) Intersect each given $L \in \text{Gr}_k(\mathbb{F}^d)$ with appropriate principal (constant) hyperplanes to obtain k lines; then proceed as in d), exploiting that the hypothesis asserts $\tilde{\mathcal{S}}$ to be the induced set of a family of subspace tuples.

- f) According to b), $f_{i,j}^{-1}[=1]$ has $*$ -constructible (and in particular $*$ -semialgebraic) set of Plücker coordinates for each $\vec{j} = (j_1, \dots, j_m)$. Hence so is, according to Lemma 61f), the set of Plücker Coordinates of $\pi_{\vec{i}}(f_{i,j}^{-1}[=1])$; and also the union, taken over all (finitely many) $\vec{i} \in \{0, 1, \dots, d\}^n$. Now this union $\bigcup_{\vec{j}} \pi_{\vec{i}}(f_{i,j}^{-1}[=1])$ coincides with \mathcal{S} .
- g) Since \mathcal{S} has $*$ -semialgebraic Plücker Coordinates $\tilde{\mathcal{S}}$, Lemma 61f) yields $\mathcal{S} = \pi_{\vec{i}}\mathcal{T}$ for some $\mathcal{T} \subseteq \text{Gr}_{i,j}(\mathbb{F}^d)$ with $\tilde{\mathcal{T}}$ $*$ -constructible. Now according to e), $\mathcal{T} = f_{i,j}^{-1}[=1]$ for some formula f with constants.
- h) In view of Fact 22 and Theorem 23, it suffices to consider the cases $d \geq 3$. According to f), the set of Plücker coordinates of $\{(\vec{X}, \vec{Y}) : \exists \vec{Z} : f(\vec{X}, \vec{Y}, \vec{Z}) = 1\}$ is $*$ -semialgebraic over \mathbb{R} ; and so is, according to Lemma 61d), the set of Plücker coordinates of its complement, $\{(\vec{X}, \vec{Y}) : \forall \vec{Z} : f(\vec{X}, \vec{Y}, \vec{Z}) \neq 1\}$. Again invoking f), also $\{\vec{X} : \exists \vec{Y} \forall \vec{Z} : f(\vec{X}, \vec{Y}, \vec{Z}) \neq 1\}$ has set of Plücker coordinates $*$ -semialgebraic over \mathbb{R} ; hence so does its complement, $\{\vec{X} : \forall \vec{Y} \exists \vec{Z} : f(\vec{X}, \vec{Y}, \vec{Z}) = 1\}$. Therefore, according to g), this set is of the claimed form. \square

5 Computability and Complexity in Indefinitely-Finite Dimensions

Note that both Theorem 43 and Theorem 44d) apply only to (arbitrary but) definite dimensions. As announced in the introduction, we shall now proceed to indefinite dimensions by extending Definition 1:

- Definition 65.** a) Write $\mathbb{F}^\infty := \ell^2(\mathbb{F})$ for the Hilbert space of square-summable sequences over either $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.
- b) For $d \in \mathbb{N}$, let $\text{Gr}_d(\mathbb{F}^\infty) := \{X \subseteq \mathbb{F}^\infty \text{ linear subspace of } \dim(X) = d\}$ and $\text{Gr}_{-d}(\mathbb{F}^\infty) := \{X \subseteq \mathbb{F}^\infty \text{ linear subspace of } \dim(\neg X) = d\}$; $\text{Gr}_{\pm d}(\mathbb{F}^\infty) := \text{Gr}_d(\mathbb{F}^\infty) \cup \text{Gr}_{-d}(\mathbb{F}^\infty)$ and $\text{Gr}(\mathbb{F}^\infty) = \bigcup_d \text{Gr}_{\pm d}(\mathbb{F}^\infty)$.
- c) Adopting from formal language theory the convention that “ $*$ ” denotes a wildcard, write $\text{Gr}(\mathbb{F}^*) := \bigcup_{d \in \mathbb{N}} \text{Gr}(\mathbb{F}^d)$ for the collection of all finite-dimensional Grassmannians over \mathbb{F} .

The subcollection $\text{Gr}_k(\mathbb{C}^\infty)$ of all k -dimensional subspaces is also known as $\text{BU}(k)$, the classifying space of the unitary group $\mathcal{U}(\mathbb{C}^k)$.

By Section 4.1, [weak] satisfiability over $\text{Gr}(\mathbb{F}^d)$ implies so over $\text{Gr}(\mathbb{F}^*)$ but not vice versa. Furthermore it is well-known [DHMW05, bottom of p.358] that satisfiability over $\text{Gr}(\mathbb{F}^*)$ implies so over $\text{Gr}(\mathcal{H})$ but not vice versa (a specific counter example is related to the **modular** law, recall Section 1.3d).

We are now in the position to repeat, and render more precisely, from [DHMW05, §4(1)] the important open

Question 66. Given a given quantum logic formula $f(\vec{X})$, is

- a) its [weak] satisfiability over $\text{Gr}(\mathbb{F}^*)$ recursively enumerable?
- b) co-recursively enumerable?
- c) How about [weak] satisfiability over $\text{Gr}(\mathbb{F}^\infty)$,
i.e. by co-/finite dimensional subspaces X_i of $\ell^2(\mathbb{F})$?
- d) How about [weak] satisfiability over the (orthomodular but non-modular) lattice of closed
(but not necessarily co-/finite dimensional) subspaces of $\ell^2(\mathbb{F})$?

It seems that d) is the most challenging aspect and beyond reach for the present work. We consider answering a-c) as intermediate goals. In fact, a) is easily answered positively: decide [weak] satisfiability of f over $\text{Gr}(\mathbb{F}^d)$ iteratively for $d = 2, 3, \dots$. And for weak satisfiability, both b) and c) turn out to have a positive answer in Theorem 68c) below. Concerning (strong) satisfiability in c), we have the surprising

Theorem 67. *A formula is satisfiable over $\text{Gr}(\mathbb{F}^\infty)$ iff it is satisfiable over $\{0, 1\}$. In particular, satisfiability over $\text{Gr}(\mathbb{F}^\infty)$ is (decidable and) \mathcal{NP} -complete.*

This sheds some new light on Example 13a). Note that a satisfying assignment \vec{X} of f in $\text{Gr}(\mathbb{F}^d)$ (i.e. with $f(\vec{X}) = \mathbb{F}^d$) does extend to $f(\bigoplus^\omega \vec{X}) = \bigoplus^\omega \mathbb{F}^d = \mathbb{F}^\infty$; however the infinite direct sum $\bigoplus^\omega \vec{X}$ has the required co-/finite dimension only in the Boolean case $\vec{X} \in \{0, 1\}$.

Proof. If $X_1, \dots, X_n \in \{0, 1\}$ is a satisfying assignment of f , then obviously $X_1, \dots, X_n \in \text{Gr}(\mathbb{F}^\infty)$ (co-/dimension 0) is also one over $\text{Gr}(\mathbb{F}^\infty)$. Conversely let $f(\vec{X}) = \mathbb{F}^\infty$ for certain $X_1, \dots, X_n \in \text{Gr}(\mathbb{F}^\infty)$, i.e. there exists $I \subseteq \{1, \dots, n\}$ such that $\dim(X_i) < \infty$ for each $i \in I$ and $\dim(\neg X_i) < \infty$ for $i \notin I$. Define $Z := \bigvee_{i \in I} X_i \vee \bigvee_{i \notin I} \neg X_i$. Then $Z \subsetneq \mathbb{F}^\infty$ contains, for all i , either X_i or $\neg X_i$; hence (Fact 4) both Z and $0 \neq \neg Z \in \text{Gr}(\mathbb{F}^\infty)$ commute with all X_i . So Lemma 36 shows $\neg Z = \Xi_{\mathbb{F}^\infty}(f; \vec{X}) \wedge \neg Z = \Xi_{\neg Z}(f; \vec{X} \wedge \neg Z)$. In other words, $\vec{X} \wedge \neg Z$ constitutes a satisfying assignment of f in $\text{Gr}(\neg Z)$; note that $X_i \wedge \neg Z = 0$ for $i \in I$ and $X_i \wedge \neg Z = \neg Z$ for $i \notin I$, that is, $X_i \wedge \neg Z \in \{0, \neg Z\}$: a Boolean assignment. \square

5.1 Weak Satisfiability over Unknown Finite Dimensions

It was easy to see that weak satisfiability over $\text{Gr}(\mathbb{F}^*)$ is recursively enumerable. In fact it is no more difficult than over $\text{Gr}(\mathbb{F}^3)$ in view of Corollary 52 and by Item c) of

Theorem 68. *Let \mathcal{H} be a Hilbert space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.*

- a) *Let formula $f(X_1, \dots, X_n)$ be weakly satisfiable over $\text{Gr}(\mathcal{H})$. Then it is also weakly satisfiable over $\text{Gr}(\mathbb{F}^{n^\ell})$ where $\ell = |f|$ denotes the syntactic length of f .*
- b) *To any $d \in \mathbb{N}$ there exists a formula of length (and Turing-computable in time) $\mathcal{O}(d \cdot \log d)$ independent of \mathbb{F} which is weakly satisfiable over $\text{Gr}(\mathbb{F}^d)$ but not over $\text{Gr}(\mathbb{F}^{d-1})$.*
- c) *Weak satisfiability over $\text{Gr}(\mathbb{C}^*)$, over $\text{Gr}(\mathbb{R}^*)$, over $\text{Gr}(\mathbb{C}^\infty)$, and over $\text{Gr}(\mathbb{R}^\infty)$ are all polynomial-time equivalent and decidable in $\text{BP}(\mathcal{NP}_{\mathbb{R}}^0)$.*

Recall that PSPACE is the best known upper bound to $\text{BP}(\mathcal{NP}_{\mathbb{R}}^0)$. We are not aware of a lower complexity bound to weak satisfiability over $\text{Gr}(\mathbb{F}^*)$. Notice the striking contrast between the quadratic upper bound due to Item a) on the least dimension of weak satisfiability on the one hand and on the other hand the exponential lower bound for the case of (strong) satisfiability in Example 41c).

A direct proof of Item a) by induction on $|f|$ fails: $f_1(\vec{X}_1) > 0$ for $\vec{X}_1 \in \text{Gr}(\mathbb{F}^{d_1})$ and $f_2(\vec{X}_2) > 0$ for $\vec{X}_2 \in \text{Gr}(\mathbb{F}^{d_2})$ does not imply $f_1(\vec{X}_1 \oplus \vec{X}_2) \wedge f_2(\vec{X}_1 \oplus \vec{X}_2) > 0$ for $\vec{X}_1 \oplus \vec{X}_2 \in \text{Gr}(\mathbb{F}^{d_1+d_2})$. Instead, we use the following

Lemma 69. *Let \mathcal{H} be a Hilbert space over \mathbb{F} .*

- a) *To a formula $f(X_1, \dots, X_n)$ of length $|f| =: \ell$, a Turing machine can construct in time polynomial in ℓ a formula $g(X_1, \dots, X_n, Y_1, \dots, Y_n)$ independent of \mathcal{H} , having $|g| \leq \ell$, devoid of negation, and such that $f(\vec{X}) = g(\vec{X}, \neg \vec{X})$ holds.*
Moreover whenever (\vec{X}, \vec{Y}) is a weak assignment of g with $Y_i \leq \neg X_i$, then \vec{X} is one of f .

- b) Let g be a formula of length ℓ without negations and $X_1, \dots, X_n \in \text{Gr}(\mathcal{H})$ be such that $g(\vec{X}) \geq Z \in \text{Gr}_1(\mathcal{H})$. Then there exist $Y_i \in \mathcal{H}$ of $\dim(Y_i) \leq \ell$ with $Y_i \leq X_i$ such that $g(\vec{Y}) \geq Z$.
- c) Referring to the restriction of a formula from Fact 37b), it holds $C(X, Y)|_{C(X, Y)} = C(X, Y)$ and $C(X, Y)|_{-C(X, Y)} = 0$.

Proof (Lemma 69).

- a) Observe that g can be obtained from f by successive application of de Morgan's law; without affecting its size. A weakly satisfying assignment \vec{X} of f obviously yields one of g via $\vec{Y} := \neg \vec{X}$. Conversely $g(\vec{X}, \vec{Y}) > 0$ with $Y_i \leq \neg X_i$ implies $f(\vec{X}) = g(\vec{X}, \neg \vec{X}) \geq g(\vec{X}, \vec{Y}) > 0$ because g without negations is monotone in \vec{Y} .
- b) The proof proceeds by induction on the length of g . Induction start $g = X_i$ is obvious: take $Y_i := Z \leq g(\vec{X}) = X_i$.
For the induction step, first consider the case $g = g_1 \wedge g_2$. By presumption, $g_1(\vec{X}) \wedge g_2(\vec{X}) \geq Z$; i.e. $g_j(\vec{X}) \geq Z$ for $j = 1, 2$. Thus, by induction hypothesis on g_j of length $|g_j| < |g|$, there exist $Y_i^{(j)} \leq X_i$ of $\dim(Y_i^{(j)}) \leq |g_j|$ such that $g_j(\vec{Y}^{(j)}) \geq Z$. Hence, $Y_i := Y_i^{(1)} \vee Y_i^{(2)} \leq X_i$ has $\dim(Y_i) \leq (|g_1| + |g_2|) \leq \ell$ and $g_j(\vec{Y}) \geq g_j(\vec{Y}^{(j)}) \geq Z$ by monotonicity: $g(\vec{Y}) \geq Z$.
It remains to consider the inductive step in case $g = g_1 \vee g_2$. By presumption, $g_1(\vec{X}) \vee g_2(\vec{X}) \geq Z$; i.e. $g_j(\vec{X}) \geq Z_j$ for appropriate $Z_1, Z_2 \in \text{Gr}_{\leq 1}(\mathcal{H})$ with $Z_1 \vee Z_2 \geq Z$. Thus, by induction hypothesis, there exist $Y_i^{(j)} \leq X_i$ of $\dim(Y_i^{(j)}) \leq |g_j|$ such that $g_j(\vec{Y}^{(j)}) \geq Z_j$. Therefore $Y_i := Y_i^{(1)} \vee Y_i^{(2)} \leq X_i$ has $\dim(Y_i) \leq \ell$ and $g_j(\vec{Y}) \geq g_j(\vec{Y}^{(j)}) \geq Z_j$ by monotonicity: $g(\vec{Y}) \geq Z$.
- e) clear from the characterization of $\mathcal{MO}_4 \times \{0, 1\}^4$ as the free modular ortholattice over two generators. \square

Proof (Theorem 68).

- a) In view of Lemma 69a), consider the negation-free formula g . By hypothesis, it admits an assignment (\vec{X}, \vec{Y}) with $Y_i \leq \neg X_i$ and $g(\vec{X}, \vec{Y}) \geq Z$ for some $\dim(Z) = 1$. Lemma 69b) yields an assignment (\vec{X}', \vec{Y}') with $X'_i \leq X_i$ and $Y'_i \leq Y_i \leq \neg X_i$ and $\dim(X'_i), \dim(Y'_i) \leq |g| = \ell$ and $g(\vec{X}', \vec{Y}') \geq Z$; hence Lemma 69a) asserts \vec{X}' to be a weakly satisfying assignment of f . Now the X'_i all live within the subspace $\bigvee_{i=1}^n X'_i$ of \mathcal{H} of dimension at most $n\ell$.
- b) Consider $\psi_{1,d}$ from Lemma 40f).
Alternatively, and attaining a length of only $\mathcal{O}(d^2)$, recall [Huhn72, MaRo87] that a modular ortholattice like $\text{Gr}(\mathbb{F}^d)$ fails n -distributivity iff it contains a non-trivial orthogonal $(n+1)$ -frame; where the latter is obviously equivalent to $d > n$. Here n -distributivity is defined by the condition

$$X_0 \wedge \bigvee_{i=1}^n X_i =: f_n(X_0, X_1, \dots, X_n) \stackrel{!}{=} \bigvee_{i=1}^n (X_0 \wedge \bigvee_{j \neq i} X_j) =: g_n(\vec{X}) .$$

Note that both f_n and g_n have length quadratic in n . Moreover, $X_0 \wedge \bigvee_{j \neq i} X_j \leq f_n(\vec{X})$ shows $g_n \leq f_n$; hence failure of n -distributivity $g_n \neq f_n$ is equivalent to $d > n$ on the one hand, and on the other hand to the weak satisfiability over $\text{Gr}(\mathbb{F}^d)$ of $f_n \wedge \neg g_n =: \psi_{n+1}$ by Example 7g).

- c) [Weak] satisfiability over $\text{Gr}(\mathbb{R}^*)$ and $\text{Gr}(\mathbb{C}^*)$ are mutually equivalent according to Proposition 42b+d); similarly for $\text{Gr}(\mathbb{R}^\infty)$ and $\text{Gr}(\mathbb{C}^\infty)$. In fact the homomorphisms $\text{Gr}(\mathbb{R}^d) \ni X \mapsto \underline{X} \in \text{Gr}(\mathbb{C}^d)$ and $\text{Gr}(\mathbb{C}^d) \ni X \mapsto \tilde{X} \in \text{Gr}(\mathbb{R}^{2d})$ from Proposition 42c+e) are easily verified to extend to $\text{Gr}(\mathbb{R}^\infty) \rightarrow \text{Gr}(\mathbb{C}^\infty)$ and $\text{Gr}(\mathbb{C}^\infty) \rightarrow \text{Gr}(\mathbb{R}^\infty)$, respectively. Any weakly satisfying assignment in $\text{Gr}(\mathbb{F}^*)$ also belongs to $\text{Gr}(\mathbb{F}^\infty)$. Conversely if f is weakly satisfiable over $\text{Gr}(\mathbb{F}^\infty)$ then, by a), so is it over $\text{Gr}(\mathbb{F}^{n \cdot |f|})$. And the latter is decidable in complexity class $\text{BP}(\mathcal{NP}_{\mathbb{R}}^0)$ by Theorem 44d). \square

Conjecture 70. *Weak satisfiability over $\text{Gr}(\mathbb{F}^*)$ is in \mathcal{NP} .*

5.2 (Strong) Satisfiability over Unknown Finite Dimensions

In the case $\text{Gr}(\mathbb{F}^*)$ of arbitrary finite dimensions, satisfiability is (as opposed to weak satisfiability) not known to be decidable. The same holds for real $*$ -polynomial feasibility for the case of arbitrary finite dimensions:

- Definition 71.** a) For p a noncommutative $*$ -polynomial admitting a real matrix root $\vec{Y} \in \mathbb{R}^{d \times d}$ for some $d \in \mathbb{N}$, denote by $\delta_{\mathbb{R}}(p)$ the least such dimension d ; $\delta_{\mathbb{R}}(p) := \perp$ otherwise.
b) *Noncommutative real $*$ -polynomial feasibility over unknown finite dimensions* is the problem of deciding whether a given noncommutative quartic $*$ -polynomial p with coefficients $\{0, \pm 1, \pm 2\}$ has $\delta_{\mathbb{R}}(p) \neq \perp$.
c) For $n \in \mathbb{N}$, let $\delta_{\mathbb{R},4}^*(n) := \max \{ \delta_{\mathbb{R}}(p) : p \text{ as in b) with } n \text{ variables has } \delta_{\mathbb{R}}(p) \neq \perp \}$.

Observe that there are at most $5^{\mathcal{O}(n^4)}$ n -variate p as in b); hence the maximum in c) exists.

Theorem 72. a) *The (strong) satisfiability problem over unknown finite dimensions is \mathcal{NP} -hard. More precisely, a formula $f(X_1, \dots, X_n)$ is classically satisfiable over Booleans $\{0, 1\}$ iff the following formula is satisfiable over $\text{Gr}(\mathbb{F}^d)$ for every (equivalently: for some) $d \in \mathbb{N}$:*

$$\tilde{f}(X_1, \dots, X_n) := f(X_1, \dots, X_n) \wedge \bigwedge_{i < j} C(X_i, X_j) .$$

- b) *Satisfiability over unknown finite dimensions is even hard for real polynomial feasibility.*
c) *Satisfiability of a given quantum logic formula over $\text{Gr}(\mathbb{R}^d)$ is polynomial-time reducible to noncommutative real $*$ -polynomial feasibility in dimension d . The reduction does not depend on d .*
d) *Satisfiability of a given quantum logic formula over $\text{Gr}(\mathbb{R}^*)$ (equivalently: over $\text{Gr}(\mathbb{C}^*)$) is polynomial-time equivalent to noncommutative real $*$ -polynomial feasibility over unknown finite dimensions.*
e) *Satisfiability of a given quantum logic formula over $\text{Gr}(\mathbb{R}^*)$ and/or $\text{Gr}(\mathbb{C}^*)$ is decidable iff the function $\delta_{\mathbb{R},4} : \mathbb{N} \rightarrow \mathbb{N}$ admits a recursive upper bound.*
f) *There exists some $c \in \mathbb{N}$ such that $\delta_{\mathbb{R},4}(n) \geq 2^{n/c}$ for all n .*

Recall that the heavily growing Ackermann function is recursive; whereas the Busy Beaver function

$$S(\ell) := \max \{ N : \text{Turing machine } M \text{ terminates after } N \text{ steps, } |\langle M \rangle| \leq \ell \}$$

admits no recursive upper bound [LePa98, PROBLEM 5.3.1].

Lemma 73. *Let \mathbb{F} denote any field and $d \in \mathbb{N}$.*

- a) For $S, T \in \mathbb{F}^{d \times d}$, it holds $\text{range}(S) \subseteq \text{range}(T) \Leftrightarrow \exists X \in \mathbb{F}^{d \times d} : S = T \cdot X$.
b) For $R, S, T \in \mathbb{F}^{d \times d}$, it holds $\text{range}(S) \vee \text{range}(T) = \text{range}(R)$ iff

$$\exists X, Y, W, Z \in \mathbb{F}^{d \times d} : R = S \cdot X + T \cdot Y \text{ and } S = R \cdot W \text{ and } T = R \cdot Z .$$

- c) For $R, S, T \in \mathbb{F}^{d \times d}$, it holds

$$\text{range}(S) = \neg \text{range}(T) \Leftrightarrow \exists X \in \mathbb{F}^{d \times d} : S \cdot T^\dagger = 0 \text{ and } (S + T) \cdot X = \text{id} .$$

Proof (Theorem 72).

- a) The mapping $f \mapsto \tilde{f}$ is obviously polynomial-time computable; hence it remains to prove the reduction property. If $\vec{X} \in \{0, 1\}$ is a satisfying assignment of f , then $C(X_i, X_j) = 1$ shows \tilde{f} to be satisfiable as well over each $\text{Gr}(\mathbb{F}^d)$. Conversely, pairwise commutativity of all X_i implies that the ortholattice generated by them is isomorphic to $\{0, 1\}^k$ for some $k \leq n$ (Fact 4d); and satisfiability of f over this $\{0, 1\}^k$ implies so over $\{0, 1\}$.
b) follows from Item d) in connection with Theorem 51.
c) We are given a formula $f(X_1, \dots, X_n)$ and shall represent each $Y \in \text{Gr}(\mathbb{R}^d)$ by any matrix $\hat{Y} \in \mathbb{R}^{d \times d}$ with $Y = \text{range}(\hat{Y})$. Lemma 73b+c) shows how to encode the connectives “ \vee ” and “ \neg ” (and by de Morgan also “ \wedge ”) in terms of matrix equations of degree 2 with constants 0, 1, -1 by introducing new existentially quantified variables. We obtain in this way a reduction (which is easily seen polynomial-time computable) to the feasibility of a system of quadratic *-polynomial equations

$$p_j(\hat{Y}_1, \dots, \hat{Y}_m) = 0, \quad j = 1, \dots, J \quad (23)$$

over $\mathbb{R}^{d \times d}$. Now a sum of matrices of the form $T_j \cdot T_j^\dagger$ is always positive semi-definite; and equal to 0 iff all $T_j = 0$. Therefore equation system (23) is equivalent to the single quartic equation $0 = p(\hat{Y}_1, \dots, \hat{Y}_m) := \sum_{j=1}^J p_j(\hat{Y}_1, \dots, \hat{Y}_m)^\dagger \cdot p_j(\hat{Y}_1, \dots, \hat{Y}_m)$.

- d) First note that satisfiability over $\text{Gr}(\mathbb{R}^*)$ and over $\text{Gr}(\mathbb{C}^*)$ are equivalent due to Proposition 42b+d). Secondly, Theorem 51c) has given a reduction from real *-polynomial feasibility of p in dimension d to satisfiability of f_p over $\text{Gr}(\mathbb{R}^{3d})$: independent of d , that is f_p is satisfiable for some dimension divisible by 3 iff p admits a root in some dimension. It remains to adjust f_p to be satisfiable only in dimensions divisible by 3; for instance by requiring additional variables W_0, W_1, W_2 of $\dim(W_0) = \dim(W_1) = \dim(W_2)$ with $0 = W_j \wedge (W_{j+1} \vee W_{j+2})$ and $1 = W_0 \vee W_1 \vee W_2$. The converse reduction is given by c).
e) First suppose $\delta_{\mathbb{R},4}(n) \leq \alpha(n)$ for some recursive $\alpha : \mathbb{N} \rightarrow \mathbb{N}$. Then, given quantum logic formula f , apply d) to reduce its satisfiability over $\text{Gr}(\mathbb{F}^*)$ to the real feasibility of some noncommutative *-polynomial $p_f(X_1, \dots, X_n)$. Next use the hypothesis to calculate $D := \alpha(n) \geq \delta_{\mathbb{R},4}(n)$ and combine Theorems 51d) and 44d) to decide whether p_f is feasible in $\mathbb{R}^{d \times d}$ for some $d = 1, \dots, D$: If so, also f is satisfiable in some dimension by construction of p_f ; whereas if not, by the definition of δ , neither p_f nor f are feasible/solvable in any finite dimension.

Conversely suppose that satisfiability of a given formula f over $\text{Gr}(\mathbb{F}^*)$ is decidable. Given $n \in \mathbb{N}$, enumerate all $5^{\mathcal{O}(n^4)}$ quartic noncommutative *-polynomials p in n variables with coefficients from $\{0, \pm 1, \pm 2\}$. For each such p , use d) to whether it is feasibility in unknown real dimensions; and if so, apply Theorem 51d) to decide iteratively for $d = 1, 2, \dots$ whether p is feasible in $\mathbb{R}^{d \times d}$: the first such d obviously yields the value $\delta_{\mathbb{R}}(p)$. Finally taking the maximum over the finitely many p , we have effectively calculated $\delta_{\mathbb{R},4}(n)$ as a recursive upper bound to itself.

- f) Use Lemma 73 to translate Example 41c) to the setting of matrices. This yields a system of quadratic equations in $\mathcal{O}(n)$ (i.e. $\leq c \cdot n$ for some $c \in \mathbb{N}$) variables feasible in dimension 2^n but not below. Composing them into one single quartic polynomial p_n yields $\delta_{\mathbb{R},4}(cn) \geq \delta_{\mathbb{R}}(p_n) = 2^n$. \square

Proof (Lemma 73).

- a) Let $\vec{s}_1, \dots, \vec{s}_d \in \mathbb{F}^d$ denote the columns of S and $\vec{t}_1, \dots, \vec{t}_d$ those of T . Now observe that $\text{range}(S) \subseteq \text{range}(T)$ is equivalent to the \vec{s}_i being linear combinations of the \vec{t}_j .
b) similarly.
c) $S \cdot T^\dagger = 0$ is equivalent to $\text{range}(S) \perp \text{range}(T)$; and $\exists X : (S + T) \cdot X = \text{id}$ means $\text{range}(S + T) = \mathbb{F}^d$. \square

The proof of Theorem 72d) in fact reveals

Scholium 74. *A noncommutative $*$ -polynomial p admits a real matrix root $\vec{Y} \in \mathbb{R}^{d \times d}$ for some $d \in \mathbb{N}$ iff it admits a complex matrix root $\vec{Y} \in \mathbb{C}^{e \times e}$ for some $e \in \mathbb{N}$. As a matter of fact, d and e can be linearly bounded by each other.*

5.3 Arithmetical Hierarchy and an Undecidable Σ_4 -Formula

[Weak] satisfiability over $\text{Gr}(\mathbb{F}^*)$ as considered above amounts to [weak] truth in the existential first-order theory of quantum logic. It is therefore natural to consider also universally quantified and mixed formulas. In fact we have already done so in Section 4.7 when showing that, over fixed dimension and using constants, universal quantifiers can be eliminated. Hence there was no point in formally introducing first-order quantum logic; yet over indefinite finite dimension, we suggest the following quantum-logic variant of the arithmetical hierarchy:

Definition 75. *a) Let $f(\vec{X}_1, \dots, \vec{X}_k)$ denote a formula in $n_1 + \dots + n_k$ variables. Then the expression*

$$\exists \vec{X}_1 \forall \vec{X}_2 \exists \vec{X}_3 \dots Q_k \vec{X}_k : f(\vec{X}_1, \dots, \vec{X}_k)$$

is called a Σ_k -formula, where Q_k denotes “ \forall ” for k even and “ \exists ” in case k is odd.

Similarly, a Π_k -formula starts with a universal quantifier followed by $k - 1$ alternating quantifiers.

- b) *This Σ_k -formula is **true** (strongly valid) over $\text{Gr}(\mathbb{F}^d)$ if there exists $\vec{X}_1 \in \text{Gr}(\mathbb{F}^d)$ such that for each $\vec{X}_2 \in \text{Gr}(\mathbb{F}^d)$ there exists \dots such that $f(\vec{X}_1, \dots, \vec{X}_k) = 1$.*

Similarly for Π_k formulas and for weak validity.

- c) *A Σ_k -formula is strongly/weakly valid over $\text{Gr}(\mathbb{F}^*)$ if it is over $\text{Gr}(\mathbb{F}^d)$ for some $d \in \mathbb{N}$.
A Π_k -formula is strongly/weakly valid over $\text{Gr}(\mathbb{F}^*)$ if it is over $\text{Gr}(\mathbb{F}^d)$ for every $d \in \mathbb{N}$.*

Note that the negation of a weakly valid Σ_k formula is a strongly valid Π_k formula; and the negation of a weakly valid Π_k formula is a strongly valid Σ_k formula.

We also emphasize that the formulas under consideration are in prenex form with matrix f formed by only quantum (but no Boolean) connectives, recall Remark 6 and Example 20. Indeed, Theorem 35 has relied essentially on the dimension to be arbitrary but fixed. Uniformly in the dimension (yet at the cost of more quantifier alternations), we have the following

Example 76. *a) For $X \in \text{Gr}(\mathbb{F}^d)$, it holds: $X \in \{0, 1\}$ iff $C(X, Y) = 1$ for all $Y \in \text{Gr}(\mathbb{F}^d)$.*

- b) There is no formula $f(X, \vec{Y})$ such that for all $d \in \mathbb{N}$ (or merely for all $d \in \{d_1, d_2, d_1 + d_2\}$ with d_1, d_2 arbitrary but fixed, not necessarily distinct) it holds: $X \in \{0, 1\} \Leftrightarrow \exists \vec{Y} \in \text{Gr}(\mathbb{F}^d) : f(X; \vec{Y}) = 1$; similarly for $f(X; \vec{Y}) \neq 0$.
- c) The three-variate formula $f(X, Y, Z) := C(X, Y) \wedge (X \vee Y) \wedge (C(X, Z) \vee C(Y, Z))$ has the property that, for each $d \in \mathbb{N}$ and each field $\mathbb{F} \subseteq \mathbb{C}$,

$$\forall X, Y \in \text{Gr}(\mathbb{F}^d) : (X = 1 \parallel Y = 1 \Leftrightarrow \forall Z \in \text{Gr}(\mathbb{F}^d) : f(X, Y, Z) = 1) .$$

- d) For $X, Y \in \text{Gr}(\mathbb{F}^d)$ it holds $\dim(X) \leq \dim(Y)$ iff

$$\exists Z : \Pi(\Pi(X, Z), X) = X \ \&\& \ \Pi(\Pi(Y, Z), Y) = Y \ \&\& \ \Pi(X, Z) \leq \Pi(Y, Z) . \quad (24)$$

- e) Over $\text{Gr}(\mathbb{F}^d)$, it holds

$$X \neq 0 \iff \exists Y \forall Z : Y \subseteq X \ \&\& \ (Z = 0 \parallel \dim(Z) \geq \dim(Y)) .$$

Proof. a) $X \in \{0, 1\}$ commutes with all Y . For the converse, suppose $0 < \dim(X) =: n < d$.

In view of Fact 4a) it suffices to construct 1D Y with $Y \not\subseteq X$ and $Y \not\subseteq \neg X$, because then $0 = (Y \wedge X) \vee (Y \wedge \neg X) = c(Y, X)$. To this end write $X = \vec{a}_1\mathbb{F} + \vec{a}_2\mathbb{F} + \dots + \vec{a}_n\mathbb{F}$ for pairwise orthogonal nonzero $\vec{a}_1, \dots, \vec{a}_n, \vec{a}_{n+1}, \dots, \vec{a}_d \in \mathbb{F}^d$. Now let $Y := (\vec{a}_1 + \vec{a}_d \langle \vec{a}_1 | \vec{a}_1 \rangle) \mathbb{F}$. Then, obviously, Y is orthogonal to $\vec{a}_2\mathbb{F} + \dots + \vec{a}_n\mathbb{F}$ and different (i.e. disjoint) from $\vec{a}_1\mathbb{F}$, hence $Y \not\subseteq X$. Similarly, $Y \not\subseteq \vec{a}_{n+1}\mathbb{F} + \dots + \vec{a}_d\mathbb{F} = \neg X$.

- b) Suppose the converse. Then there exist $\vec{V} \in \text{Gr}(\mathbb{F}^{d_1})$ and $\vec{W} \in \text{Gr}(\mathbb{F}^{d_2})$ with $f(\{0^{d_1}\}, \vec{V}) = \mathbb{F}^{d_1}$ and $f(\mathbb{F}^{d_2}, \vec{W}) = \mathbb{F}^{d_2}$. Hence in dimension $d_1 + d_2$, $X := \{0^{d_1}\} \times \mathbb{F}^{d_2}$ has $f(X, \vec{V} \times \vec{W}) = \mathbb{F}^{d_1+d_2} = 1$ by Lemma 36, thus contradicting $X \notin \{0^{d_1+d_2}, \mathbb{F}^{d_1+d_2}\}$.

Similarly, $f(0^{d_1}, \vec{V}) \neq 0$ implies $f(X, \vec{V} \times \vec{W}) \neq 0$.

- c) If $X = 1$, then obviously $X \vee Y = 1 = C(X, Y)$ and $C(X, Z) = 1$ for all Z ; similarly for $Y = 1$. For the converse implication, consider $X, Y \in \text{Gr}(\mathbb{F}^d)$; by Lemma 40a) w.l.o.g. $X = \mathbb{F}^{k+\ell} \times 0^m$ with $\dim(Y) = k + m$ and $k + \ell + m = d$. $C(X, Y) = 1$ implies $Y = Y_1 \times Y_2$ with $Y_1 \subseteq \mathbb{F}^{k+\ell}$ and $Y_2 \subseteq \mathbb{F}^m$; in fact $Y_2 = \mathbb{F}^m$ because of $X \vee Y = 1$. Again invoking Lemma 40a), w.l.o.g. $Y_1 = \mathbb{F}^k$. Also w.l.o.g. $\ell \leq m$: otherwise exchange X, Y . Thus

$$\begin{aligned} X &= \mathbb{F}^k \times \mathbb{F}^\ell \times 0^\ell \times 0^{m-\ell}, & \neg X &= 0^k \times 0^\ell \times \mathbb{F}^\ell \times \mathbb{F}^{m-\ell}, \\ Y &= \mathbb{F}^k \times 0^\ell \times \mathbb{F}^\ell \times \mathbb{F}^{m-\ell}, & \neg Y &= 0^k \times \mathbb{F}^\ell \times 0^\ell \times 0^{m-\ell}. \end{aligned}$$

Now let $Z := \mathbb{F}^k \times \{(\vec{x}, \vec{x}) : \vec{x} \in \mathbb{F}^\ell\} \times \mathbb{F}^{m-\ell}$, i.e. $\neg Z = 0^k \times \{(\vec{x}, -\vec{x}) : \vec{x} \in \mathbb{F}^\ell\} \times 0^{m-\ell}$. It is easy to verify

$$\begin{aligned} X \wedge Z &= \mathbb{F}^k \times 0^\ell \times 0^\ell \times 0^{m-\ell}, & X \wedge \neg Z &= 0^k \times 0^\ell \times 0^\ell \times 0^{m-\ell}, \\ Y \wedge Z &= \mathbb{F}^k \times 0^\ell \times 0^\ell \times \mathbb{F}^{m-\ell}, & Y \wedge \neg Z &= 0^k \times 0^\ell \times 0^\ell \times 0^{m-\ell}, \\ \neg X \wedge Z &= 0^k \times 0^\ell \times 0^\ell \times \mathbb{F}^{m-\ell}, & \neg X \wedge \neg Z &= 0^k \times 0^\ell \times 0^\ell \times 0^{m-\ell}, \\ \neg Y \wedge Z &= 0^k \times 0^\ell \times 0^\ell \times \mathbb{F}^{m-\ell}, & \neg Y \wedge \neg Z &= 0^k \times 0^\ell \times 0^\ell \times 0^{m-\ell}. \end{aligned}$$

Thus $C(X, Z) = C(Y, Z) = \mathbb{F}^k \times 0^\ell \times 0^\ell \times \mathbb{F}^{m-\ell}$ equates to 1 iff $\ell = 0$, i.e. iff $Y = 1$.

- d) Note that $\Pi(\Pi(X, Z), X) = X$ requires $\dim(\Pi(X, Z)) = \dim(X)$ by Fact 37c); similarly $\dim(\Pi(Y, Z)) = \dim(Y)$. Hence Equation (24) implies $\dim(X) \leq \dim(Y)$.

- i) For the converse, first consider the case $C(X, Y) = 1$. As in a) w.l.o.g. $X = \mathbb{F}^k \times \mathbb{F}^\ell \times 0^a \times 0^b$ and $Y = \mathbb{F}^k \times 0^\ell \times \mathbb{F}^a \times 0^b$ with $k, \ell, a, b \in \mathbb{N}$; $\ell \leq a$ since $\dim(X) \leq \dim(Y)$. Then $Z := Z_1(X, Y) := \mathbb{F}^k \times \{(x_1, \dots, x_\ell; x_1, \dots, x_\ell, x_{\ell+1}, \dots, x_a) : x_i \in \mathbb{F}\} \times 0^b$ has $\neg Z = 0^k \times \{(x_1, \dots, x_\ell; -x_1, \dots, -x_\ell, 0, \dots, 0) : x_i \in \mathbb{F}\} \times \mathbb{F}^b$, hence Equation (24) is readily verified.
- ii) Now consider the case $C(X, Y) = 0$. Then necessarily $\dim(X) = \dim(Y) = d/2$ and $\Pi(X, Y) = Y$ and $\Pi(Y, X) = X$; so $Z := Z_0(X, Y) := X$ will do (as well as $Z := Y$, of course).
- iii) Finally, in the general case, let $Z := Z_1 \vee Z_0$ with $Z_1 := Z_1(X, Y)|_{C(X, Y)} \subseteq C(X, Y)$ and $Z_0 := Z_0(X, Y)|_{\neg C(X, Y)} \subseteq \neg C(X, Y)$ as in i+ii). Then Z commutes with $C(X, Y)$; and so do X and Y (Lemma 69c). Thus, according to Lemma 36, Equation (24) holds not only restricted to $C(X, Y)$ (i) and to $\neg C(X, Y)$ (ii), but also on their sum, i.e. on the whole space.
- e) In case $X \neq 0$, consider one-dimensional $Y \subseteq X$. Then any $Z \neq 0$ has $\dim(Z) \geq 1 = \dim(Y)$. Whereas in case $X = 0$, any $Y \subseteq X$ must be $Y = 0$; hence 1D Z violates the condition. \square

Example 76a+b) reveals that universal quantification strictly adds to the expressiveness:

Corollary 77. *Quantum logic over $\text{Gr}(\mathbb{F}^*)$ is not model complete.*

Theorem 78. *Strong validity over $\text{Gr}(\mathbb{C}^*)$ of a given Σ_4 -formula is generally undecidable to a Turing machine.*

We point out the analogy to the historical [Robi49, THEOREM 3.1], showing the undecidability of the first-order theory of the rationals (recall Fact 45f) by defining integers (Fact 45e) within \mathbb{Q} via a Π_4 -formula.

Our proof of Theorem 78 is based on the following

Fact 79. *Let $M(\vec{x}) := \prod_{j=1}^m x_{i_j}$ denote a noncommutative monomial in variables x_1, \dots, x_n and let $E(\vec{x})$ denote an equation of two such monomials. More generally, consider finitely many such equations $E_0(\vec{x}), E_1(\vec{x}), \dots, E_k(\vec{x})$ and, for a field \mathbb{F} , the quantified formula*

$$\exists d \in \mathbb{N} \exists x_1, \dots, x_n \in \mathbb{F}^{d \times d} : E_1(\vec{x}) \wedge \dots \wedge E_k(\vec{x}) \wedge \neg E_0(\vec{x}) \quad (25)$$

- a) *Over $\mathbb{F} := \mathbb{F}_2 = \{0, 1\}$ and upon input of (the monomials constituting) E_0, E_1, \dots, E_k , the validity of (25) is semidecidable*
- b) *but not decidable to a Turing machine.*
- c) *The validity of (25) is independent of the field \mathbb{F} under consideration.*

Item a) follows by enumerating all $d \in \mathbb{N}$ and all $x_1, \dots, x_n \in \{0, 1\}^{d \times d}$. For Item b), recall that (the complement of) the word problem for finite semigroups is undecidable to a Turing machine, cf. [Gure66] and see also [ABR92]. Now [Lips74, SECTION 3] shows that (25) is equivalent to the following:

There exists a finite semigroup S and $x_1, \dots, x_n \in S$ such that $E_1(\vec{x}) \wedge \dots \wedge E_k(\vec{x}) \wedge \neg E_0(\vec{x})$ holds.

The argument proceeds as follows: Note that the multiplicative semigroup $\mathbb{F}_2^{d \times d}$ is finite. And, conversely, any finite semigroup S is contained (namely via the right-regular representation) in the multiplicative semigroup $\mathbb{F}_2^{d \times d}$ for $d := \text{Card}(S)$. Hence an instance

$$\forall \text{ finite semigroups } S \quad \forall x_1, \dots, x_n \in S : \quad \bigwedge_{i=1}^k E_i(\vec{x}) \Rightarrow E_0(\vec{x})$$

of the restricted word problem for finite semigroups is refutable iff (25) holds. For Item c), refer to [Lips74, LEMMA 3.5]. There it is shown that, based on the compactness theorem, the validity of (25) over the field of algebraic numbers implies the same over some finite algebraic extension of a finite field. (This is sometimes called **Robinson's Principle**...) \square

Proof (Theorem 78). We already know from Corollary 49 how to embed the matrix ring $\mathbb{F}^{d \times d}$ ($\mathbb{F} = \mathbb{R}, \mathbb{C}$) into $\text{Gr}(\mathbb{C}^{3d})$, uniformly in d ; and $\mathbb{F}^{* \times *}$ into $\text{Gr}(\mathbb{C}^*)$ (Theorem 72d). This embedding translates satisfiable (conjunctions of) equations $E_i(\vec{X})$ over $\mathbb{F}^{* \times *}$ into strongly satisfiable equations $\tilde{E}_i(\vec{Y}) = 1$ over $\text{Gr}(\mathbb{F}^*)$. The difficulty is with the inequality “ $\neg E_0(\vec{X})$ ” in (25) translated to “ $\tilde{E}_0(\vec{Y}) \neq 1$ ”. But, combining Example 76c-e), this inequality over $\text{Gr}(\mathbb{F}^*)$ can be expressed as equality (strong validity) of a Σ_4 -formula; which, when combined with the quantifiers in (25) remains a Σ_4 -formula. \square

5.4 Fractional Satisfiability: a Threshold Between Weak and Strong

Recall that formula $f(\vec{X})$ is satisfiable over $\text{Gr}(\mathbb{F}^*)$ iff $\exists d \in \mathbb{N} : \text{maxdim}(f, d) \geq d =: \vartheta_s(d)$; and weakly satisfiable over $\text{Gr}(\mathbb{F}^*)$ iff $\exists d \in \mathbb{N} : \text{maxdim}(f, d) \geq 1 =: \vartheta_w(d)$.

Observation 80. a) Both ϑ_s and ϑ_w are subadditive functions $\vartheta : \mathbb{N} \rightarrow \mathbb{N}$ i.e. satisfy $\vartheta(d_1 + d_2) \leq \vartheta(d_1) + \vartheta(d_2)$.
b) The function $0 \leq x \mapsto \lceil x \rceil \in \mathbb{N}$ is subadditive.
c) Let $p, q \in \mathbb{N}$ and $0 < \epsilon < 1$. Then the functions $d \mapsto \lceil d/q \rceil \cdot p$ and $d \mapsto \lceil d^{1-\epsilon} \rceil$ are subadditive.

Indeed, $\mathbb{N} \ni \lceil x \rceil + \lceil y \rceil \geq x + y$ implies that $\lceil x + y \rceil$, the least integer $\geq x + y$, is $\leq \lceil x \rceil + \lceil y \rceil$ as claimed in b). Thus c) follows regarding that subadditive functions (like $0 \leq x \mapsto x^{1-\epsilon}$) are closed under composition. This suggests the following generalization:

Definition 81. For a function $\vartheta : \mathbb{N} \rightarrow \mathbb{N}$ with $\vartheta(d) \leq d$, call a formula $f(\vec{X})$ ϑ -satisfiable iff $\exists d \in \mathbb{N} : \text{maxdim}(f, d) \geq \vartheta(d)$.

Now weak (i.e. ϑ_w -) satisfiability is decidable, in fact in complexity class $\text{BP}(\mathcal{NP}_{\mathbb{R}}^0)$; whereas (strong, i.e. ϑ_s -) satisfiability is semi-decidable but not known decidable. Theorem 82b-d) explores the border between both extremes in terms of further functions ϑ .

Theorem 82. Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

- For $\vartheta : \mathbb{N} \rightarrow \mathbb{N}$, let $\tilde{\vartheta}(d) := \min\{\vartheta(e) : e \geq d\}$. Then $\tilde{\vartheta}$ is nondecreasing; and subadditive if ϑ is. Moreover, a formula $f(\vec{X})$ is $\tilde{\vartheta}$ -satisfiable iff it is ϑ -satisfiable.
- Let $\vartheta : \mathbb{N} \rightarrow \mathbb{N}$ be recursive. Then ϑ -satisfiability over $\text{Gr}(\mathbb{F}^*)$ is semi-decidable. If ϑ is furthermore subadditive with $0 = \inf_d \vartheta(d)/d$, then ϑ -satisfiability over $\text{Gr}(\mathbb{F}^*)$ of a given formula f is decidable.
- Suppose ϑ is polynomial-time computable and $\vartheta(d) \leq \mathcal{O}(d^{1-\epsilon})$ (i.e. $\{\vartheta(d)/d^{1-\epsilon} : d \in \mathbb{N}\}$ be bounded) for some $\epsilon > 0$. Then ϑ -satisfiability is in $\text{BP}(\mathcal{NP}_{\mathbb{R}}^0)$.

d) Let $p \leq q \in \mathbb{N}$ and $\vartheta(d) := \lceil d/q \rceil \cdot p$. Then ϑ -satisfiability is polynomial-time equivalent to satisfiability.

Proof. a) Since $\{\vartheta(e) : e \geq d+1\} \subseteq \{\vartheta(e) : e \geq d\}$, $\tilde{\vartheta}(d+1) \geq \tilde{\vartheta}(d)$ is nondecreasing. Moreover, subadditivity of ϑ yields $\tilde{\vartheta}(d_1 + d_2) =$

$$\begin{aligned} &= \min\{\vartheta(e_1 + e_2) : e_1 + e_2 \geq d_1 + d_2\} \leq \min\{\vartheta(e_1) + \vartheta(e_2) : e_1 + e_2 \geq d_1 + d_2\} \\ &\leq \min\{\vartheta(e_1) + \vartheta(e_2) : e_1 \geq d_1, e_2 \geq d_2\} = \tilde{\vartheta}(d_1) + \tilde{\vartheta}(d_2) . \end{aligned}$$

Finally $\tilde{\vartheta}(d) \leq \vartheta(d)$ reveals that ϑ -satisfiability implies $\tilde{\vartheta}$ -satisfiability. Conversely $\maxdim(f, d) \geq \tilde{\vartheta}(d) = \vartheta(e)$ for some $e \geq d$ requires $\maxdim(f, e) \geq \maxdim(f, d)$ by Lemma 40c).

b) For each $d = 1, 2, \dots$ calculate both $\maxdim(f, d)$ (Theorem 44e) and $\vartheta(d)$ (hypothesis) and compare their values: this yields an algorithm semi-deciding ϑ -satisfiability of f .

By Fekete's Lemma, subadditivity of ϑ implies $\lim_d \vartheta(d)/d = \inf_d \vartheta(d)/d = 0$. Hence, for a given formula f in $n \leq |f|$ variables, $0 = \lim_m \vartheta(|f|^2 \cdot m)/(|f|^2 \cdot m)$. Thus there exists (in fact infinitely many) m with $\vartheta(|f|^2 \cdot m)/(|f|^2 \cdot m) \leq 1/|f|^2$, i.e. $\vartheta(|f|^2 \cdot m) \leq m$; and a Turing machine can find such m by computing $\vartheta(|f|^2 \cdot m)$ iteratively for $m = 1, 2, \dots$. Then apply Theorem 44e) to calculate $\maxdim(f, |f|^2 \cdot m)$ and compare it with $\vartheta(|f|^2 \cdot m)$: If $\maxdim(f, d) \geq \vartheta(d) \geq 1$ for some d , then $\maxdim(f, |f|^2) \geq 1$ by Theorem 68a) and $\maxdim(f, |f|^2 \cdot m) \geq m \geq \vartheta(|f|^2 \cdot m)$ by Lemma 40c).

c) Let $\mathbb{N} \ni C \geq \vartheta(d)/d^{1-\epsilon}$ for all d . Then it holds $\vartheta(d)/d \leq C/d^\epsilon$ and in particular $\vartheta(|f|^2 \cdot m)/(|f|^2 \cdot m) \leq C/(|f|^2 \cdot m)^\epsilon \leq 1/|f|^2$ for $m := \lceil C^{1/\epsilon} \cdot |f|^{2\epsilon-2} \rceil$ bounded by a polynomial in the length $|f|$ of the input. Now proceed as in b).

d) We first show polynomial-time reducibility from satisfiability to ϑ -satisfiability: Given formula f as instance of (strong) satisfiability, construct the formula $\tilde{f} := \psi_{p,q}|_f$ according to Fact 37b) and Lemma 40f), i.e. such that, for all $d \in \mathbb{N}$, $\maxdim(\tilde{f}, d) = p \cdot \lfloor \maxdim(f, d)/q \rfloor$. If the latter is $\geq p \cdot \lceil d/q \rceil$, this means (q divides d and) $\maxdim(f, d) = d$, i.e. ϑ -satisfiability of \tilde{f} requires satisfiability of f . Conversely, satisfiability of f in dimension d implies $\maxdim(f, qd) = qd$ (Lemma 40d) and therefore $\maxdim(\tilde{f}, qd) = pd = \vartheta(d)$: ϑ -satisfiability of \tilde{f} in dimension qd . It is easy to see that, since p, q are fixed, the mapping $f \mapsto \psi_{p,q}|_f$ is polynomial-time computable.

For the reverse reduction, let $f(\vec{X})$ be a given instance for ϑ -satisfiability and construct from it, again in time polynomial in the input length, the formula $\tilde{f}(\vec{X}; Y_1, \dots, Y_q)$ encoding (using Example 7g) the conjunction of the following conditions:

$$\dim(Y_i) = \dim(Y_j) \quad \forall i, j, \quad 0 = Y_i \wedge \bigvee_{j \neq i} Y_j, \quad 1 = \bigvee_{j=1}^q Y_j, \quad f(\vec{X}) \geq \bigvee_{j=1}^p Y_j .$$

Then any satisfying assignment (\vec{X}, \vec{Y}) of \tilde{f} in some dimension d has $d = q \cdot \dim(Y_j)$ and $\dim f(\vec{X}) \geq p \cdot \dim(Y_j) = \vartheta(d)$, i.e. satisfiability of \tilde{f} implies ϑ -satisfiability of f . Conversely suppose f is ϑ -satisfiable in some dimension d . Letting $\tilde{d} := q \cdot \lceil d/q \rceil \geq d$ denote the next integral multiple of q , it follows $\maxdim(f, \tilde{d}) \geq \maxdim(f, d) \geq \vartheta(d) = p \cdot \tilde{d}/q$; and there exist $\vec{X} \in \text{Gr}(\mathbb{F}^{\tilde{d}})$ and pairwise disjoint $Y_1, \dots, Y_q \in \text{Gr}_{\tilde{d}/q}(\mathbb{F}^{\tilde{d}})$ with $\dim f(\vec{X}) = \maxdim(f, \tilde{d}) = Y_1 \vee \dots \vee Y_p$, i.e. a satisfying assignment of \tilde{f} . \square

5.5 Dimensions of Satisfiability and Nonuniform Computational Complexity

Note that, by virtue of Beran's classification, any two-variate formula f satisfiable over $\text{Gr}(\mathbb{C}^*)$ is also satisfiable over $\text{Gr}(\mathbb{Q}^2)$. However, starting with three variables as in Example 15, the

first dimension which any formula satisfiable over $\text{Gr}(\mathbb{F}^*)$ is satisfiable in, is unbounded by Item c) of the following

Example 83. Let $d \in 2\mathbb{N}$ and $\mathbb{F} \subseteq \mathbb{R}$ a field.

- a) Let $A := (1, 1, 0, 0, \dots, 0, 0)\mathbb{F} + (0, 0, 1, 1, 0, 0, \dots, 0, 0)\mathbb{F} + \dots + (0, 0, \dots, 0, 0, 1, 1)\mathbb{F} \in \text{Gr}_{d/2}(\mathbb{F}^d)$, $B := (1, 0, 0, 0, 0, \dots, 0, 0, 0)\mathbb{F} + (0, 1, 1, 0, 0, \dots, 0, 0, 0)\mathbb{F} + (0, 0, 0, 1, 1, \dots, 0, 0, 0)\mathbb{F} + (0, 0, 0, \dots, 1, 1, 0)\mathbb{F} \in \text{Gr}_{d/2}(\mathbb{F}^d)$, and recall from Example 15 the projection formula $\Pi_Z(X)$. Then, for $u_1, \dots, u_d \in \mathbb{F}$, it holds

$$\begin{aligned} & \Pi_A((u_1, u_2, u_3, u_4, \dots, u_d)\mathbb{F}) \\ &= (u_1 + u_2, u_1 + u_2, u_3 + u_4, u_3 + u_4, \dots, u_{d-1} + u_d, u_{d-1} + u_d)\mathbb{F} \\ & \Pi_B((u_1, u_2, u_3, \dots, u_{d-2}, u_{d-1}, u_d)\mathbb{F}) \\ &= (2u_1, u_2 + u_3, u_2 + u_3, u_4 + u_5, u_4 + u_5, \dots, u_{d-2} + u_{d-1}, u_{d-2} + u_{d-1}, 0)\mathbb{F}. \end{aligned}$$

- b) Let A, B as in a) and $X_1 := (1, 0, 0, \dots, 0)\mathbb{F}$. Then $X_{2i+2} := \Pi_A(X_{2i+1})$ and $X_{2i+1} := \Pi_B(X_{2i})$ defines a sequence of 1D subspaces with $\bigvee_{j=1}^d X_j = \mathbb{F}^d$.

- c) Abbreviate $\Pi^{(1)}(X; A, B) := X$, $\Pi^{(2i+2)}(X; A, B) := \Pi_A(\Pi^{(2i+1)}(X; A, B))$, and $\Pi^{(2i+1)}(X; A, B) := \Pi_B(\Pi^{(2i)}(X; A, B))$. The following formula in variables X, A, B has length $\mathcal{O}(d^3)$ and is

$$\bigvee_{j=1}^d \Pi^{(j)}(X; A, B) \wedge \bigwedge_{j,\ell=1}^d \left(\Pi^{(j)}(X; A, B) \vee \neg \Pi^{(\ell)}(X; A, B) \right) \wedge \bigwedge_{j=1}^d \left(\neg \Pi^{(j)}(X; A, B) \vee \neg \bigvee_{\ell < j} \Pi^{(\ell)}(X; A, B) \right)$$

Proof. a) Observe $\neg A = (1, -1, 0, 0, \dots, 0, 0)\mathbb{F} + (0, 0, 1, -1, 0, 0, \dots, 0, 0)\mathbb{F} + \dots + (0, 0, \dots, 0, 0, 1, -1)\mathbb{F}$ and $\neg B = (0, 1, -1, 0, 0, \dots, 0, 0, 0)\mathbb{F} + (0, 0, 0, 1, -1, \dots, 0, 0, 0)\mathbb{F} + \dots + (0, 0, 0, \dots, 1, -1, 0)\mathbb{F} + (0, 0, 0, \dots, 0, 0, 1)\mathbb{F}$. Write $X := (u_1, u_2, \dots, u_d)\mathbb{F}$. Since $\Pi_A(X) = (X \vee \neg A) \wedge A$, $\vec{x} = (x_1, x_2, \dots, x_d) \in \Pi_A(X) \subseteq X \vee \neg A$ has $x_1 = su_1 + t$ and $x_2 = su_2 - t$ for some $t \in \mathbb{F}$. Moreover $\vec{x} \in A$ requires $x_1 = x_2$, i.e. $t = (su_2 - su_1)/2$ and $x_1 = x_2 = s/2(u_1 + u_2)$. Similarly $x_3 = x_4 = s/2(u_3 + u_4)$, and so on. $\Pi_B(X)$ is calculated analogously.

- b) Let $k_j := \max\{i : \exists \vec{x} \in X_j : x_i \neq 0\}$. Obviously $k_1 = 1$. From a) it follows by induction that k_{2i+2} is even and equal to $k_{2i+1} + 1$; whereas k_{2i+1} is odd and equal to $k_{2i} + 1$. Therefore (vectors spanning, respectively) X_1, X_2, \dots, X_d form an upper triangular matrix with non-zero diagonal, i.e. has rank d .

- c) Note that $\text{length}(\Pi^{(i)}(X; A, B)) \leq \mathcal{O}(i)$, hence the claimed total length bound holds. The subspaces $X_1, A, B \in \text{Gr}(\mathbb{F}^d)$ from b) yield a satisfying assignment.

Conversely suppose $X, A, B \in \text{Gr}(\mathbb{F}^k)$ satisfy the formula. Abbreviate $X_i := \Pi^{(i)}(X; A, B)$. The second conjunction requires $\dim(X_j) = \dim(X_\ell)$. Condition three says $0 = X_j \wedge \bigvee_{\ell < j} X_\ell$, hence $\dim(Y \vee Z) = \dim(Y) + \dim(Z) - \dim(Y \wedge Z)$ yields $\dim(\bigvee_{\ell=1}^j X_\ell) = \dim(\bigvee_{\ell=1}^{j-1} X_\ell) + \dim(X_j) - \dim(X_j \wedge \bigvee_{\ell=1}^{j-1} X_\ell) = \sum_{\ell=1}^j \dim(X_\ell)$ by induction on j . From the first conjunction, $\dim(\bigvee_{i=1}^d X_i) = k$: altogether $k = d \cdot \dim(X_j)$. \square

Recall that weak satisfiability over $\text{Gr}(\mathbb{F}^*)$ is at most as hard to decide as [weak] satisfiability over $\text{Gr}(\mathbb{F}^d)$ for fixed $d \geq 3$. The key to this claim was Theorem 68a), showing that every formula weakly satisfiable in *some* dimension is also weakly satisfiable in some ‘small’ dimension. Satisfiability over $\text{Gr}(\mathbb{F}^*)$ may perhaps be even undecidable. Here the crucial question is of whether, given a formula f , one can effectively derive an upper bound on the first dimension which f is satisfiable in; cmp. Theorem 72e). *Nonuniformly* in the formula f , however, deciding which dimensions f is [weakly] satisfiable in turns out as surprisingly easy.

Theorem 84. For subadditive $\vartheta : \mathbb{N} \rightarrow \mathbb{N}$ and a formula f , let $\text{Dim}_\vartheta(f) := \{n \in \mathbb{N} : \text{maxdim}(f, n) \geq \vartheta(n)\}$ denote the set of dimensions which f is ϑ -satisfiable in.

- a) $\text{Dim}_\vartheta(f)$ is an additive semigroup, that is $x, y \in \text{Dim}_\vartheta(f)$ implies $x + y \in \text{Dim}_\vartheta(f)$.
- b) Let $\emptyset \neq D \subseteq \mathbb{N}$ be an additive semigroup. Then D is “eventually an arithmetic progression”, i.e. there exist $N, d \in \mathbb{N}$ such that $D \cap [N, \infty) = (d \cdot \mathbb{N}) \cap [N, \infty)$.
- c) $\text{Dim}_\vartheta(f)$ is decidable in \mathcal{NC}^1 .

Recall that $\mathcal{NC}^1 \subseteq \mathcal{L} \subseteq \mathcal{P}$ is the class of problems solvable in logarithmic parallel time; whereas \mathcal{P} -hard problems (w.r.t. **logspace**-reductions) presumably do not admit such an efficient parallelization.

Proof. a) Observe that $\text{maxdim}(f, x) \geq \vartheta(x)$ and $\text{maxdim}(f, y) \geq \vartheta(y)$ imply $\text{maxdim}(f, x + y) \geq \text{maxdim}(f, x) + \text{maxdim}(f, y) \geq \vartheta(x) + \vartheta(y) \geq \vartheta(x + y)$ because $\text{maxdim}(f, \cdot)$ is superadditive (Lemma 40d) and ϑ subadditive by prerequisite.

b) Let $d := \lim_M \text{gcd}(D \cap \{1, 2, \dots, M\})$ denote the ‘eventual’ common denominator of D . Since $\text{gcd}(D \cap \{1, 2, \dots, M\})$ is nonincreasing, there exists some M such that $d = \text{gcd}(D \cap \{1, 2, \dots, M\}) = \text{gcd}(x_1, \dots, x_K)$ for appropriate $x_1, \dots, x_K \in D$. By definition, d divides each $x \in D$; hence one claimed inclusion already holds—independent of N , which we are still free to choose in order to satisfy the converse inclusion. To this end write $d = \sum_{k=1}^K z_k x_k$ with $z_1, \dots, z_K \in \mathbb{Z}$ according to the extended Euclidean algorithm. We show $N + nd \in D$ for all $n \in \mathbb{N}$ where $N := (x_1 - 1) \cdot \sum_{k: z_k < 0} |z_k| \cdot x_k$. Note that N is a nonnegative integer combination of x_1, \dots, x_K , hence in D . Since $n = (n \bmod x_1) + x_1 \cdot (n \text{ div } x_1)$, it suffices to demonstrate $N + nd \in D$ for $n < x_1$ because $x_1 \cdot (n \text{ div } x_1) \cdot d \in D$, trivially. Indeed,

$$\begin{aligned} N + nd &= (x_1 - 1) \cdot \sum_{k: z_k < 0} |z_k| \cdot x_k + n \cdot \sum_k z_k \cdot x_k \\ &= \sum_{k: z_k < 0} \underbrace{(x_1 - 1 - n) \cdot |z_k|}_{\geq 0} \cdot x_k + \sum_{k: z_k \geq 0} \underbrace{n \cdot z_k}_{\geq 0} \cdot x_k \in D. \end{aligned}$$

- c) In view of a) and b) it remains to decide, given $x \geq N$, whether d divides x . This can be performed in \mathcal{NC}^1 [CDL01]. The answers to the finitely many cases $x < N$ can be precomputed and thus take only constant time. \square

A Remark on Irreducible Tuples of Subspaces

Here we are interested in a kind of converse to Lemmas 36 and 40c).

Definition 85. Call n -tuple (X_1, \dots, X_n) with $X_i \in \text{Gr}(\mathbb{F}^d)$ reducible if there exists $Z \in \text{Gr}(\mathbb{F}^d) \setminus \{0, 1\}$ such that $C(X_i, Z) = 1$ holds for all $i = 1, \dots, n$; otherwise \vec{X} is irreducible. The case $d = 1$ shall also count as irreducible.

Reducibility thus means that the sub-ortholattice generated by Z, X_1, \dots, X_n is isomorphic to the direct product $[0, Z] \times [0, \neg Z]$ via the projection $X \mapsto X \wedge Z$. Irreducibility corresponds to what in group theory is called *simple*.

Theorem 86. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

- a) Suppose f is [weakly] satisfiable over $\text{Gr}(\mathbb{F}^d)$. Then, for some $e \leq d$, there exists also an irreducible $\tilde{Y} \in \text{Gr}(\mathbb{F}^e)$ [weakly] satisfying f .
- b) For $d \geq 3$, every tuple (A, B) is reducible over $\text{Gr}(\mathbb{F}^d)$.
- c) To generic $A, B \in \text{Gr}_d(\mathbb{F}^{2d})$, there exist either continuously many $Z \in \text{Gr}_{2k}(\mathbb{F}^{2d})$ satisfying $C(A, Z) = 1 = C(B, Z)$, or (namely in the ‘general’ case) precisely $\binom{d}{k}$ different such Z .
- d) To each d , there exists an irreducible d -tuple $X_1, \dots, X_d \in \text{Gr}_1(\mathbb{F}^d)$.
To each $d \geq 2$, there exists an irreducible triple $A, B, C \in \text{Gr}_d(\mathbb{F}^{2d})$.

Item c) may be rephrased as some (not necessarily Schubert-) cycle [Harr93, EXAMPLE 11.42] of $\prod^4 \text{Gr}_d(\mathbb{F}^{2d})$ with a certain orthogonality restriction. Note that $Z := \bigvee_i X_i$ always commutes with all X_i ; i.e. $X_1, \dots, X_n \in \text{Gr}(\mathbb{F}^d)$ is necessarily reducible a tuple in case $\dim(\bigvee_i X_i) < d$. Concerning Item b), recall (Section 2.2) that the free modular ortholattice on two generators is isomorphic to $\mathcal{MO}_4 \times \{0, 1\}^4$; hence the sub-ortholattice generated by $A, B \in \text{Gr}(\mathbb{F}^d)$ must be isomorphic to a direct product of zero or one factor \mathcal{MO}_4 and zero to four factors $\{0, 1\}$. In any case it can be embedded into $\text{Gr}(\mathbb{F}^2)$ and thus is reducible in $\text{Gr}(\mathbb{F}^3)$. The second part of Item d) can be shown using irreducible quadruples of defect -1 [Herr82]. In either case, we give more pedestrian and self-contained proofs below; but let us first extend Example 47a+b) as follows:

Lemma 87. *Let \mathbb{F} denote any subfield of \mathbb{C} and $d, e \in \mathbb{N}$.*

- a) Let $T \in \mathbb{F}^{d \times e}$ and $A := \text{graph}(0)$, $B := \text{graph}(T)$. Moreover let $V \in \text{Gr}(\mathbb{F}^d)$ and $W \in \text{Gr}(\mathbb{F}^e)$ with $T \cdot V \subseteq W$ and $T^\dagger \cdot W \subseteq V$. Then $X := V \oplus W \in \text{Gr}(\mathbb{F}^{d+e})$ satisfies

$$(A \wedge X) \vee (\neg A \wedge X) = X = (B \wedge X) \vee (\neg B \wedge X) . \quad (26)$$
- b) Conversely suppose that $X \in \text{Gr}(\mathbb{F}^{d+e})$ satisfies Equation (26) with A, B, T as above. Then there exist $V \in \text{Gr}(\mathbb{F}^d)$ and $W \in \text{Gr}(\mathbb{F}^e)$ with $T \cdot V \subseteq W$ and $T^\dagger \cdot W \subseteq V$ such that $X = V \oplus W$.
- c) For $d \geq 2$, there exist $S, T \in \text{GL}(\mathbb{F}^d)$ such that $T^\dagger \cdot T$ and $S^\dagger \cdot S$ have no nontrivial invariant subspace in common.

Note that in a+b), V is necessarily an invariant subspace of $T^\dagger \cdot T$; and W one of $T \cdot T^\dagger$.

Proof. a) First observe $X \wedge A = V \oplus 0^e$ and $X \wedge \neg A = 0^d \oplus (T \cdot V)$, hence $X = (X \wedge A) \vee (X \wedge \neg A)$. Next, $T^\dagger \cdot T$ is positive semidefinite. In particular, it does not have -1 as eigenvalue, i.e. $\text{id} + T^\dagger \cdot T : \mathbb{F}^d \rightarrow \mathbb{F}^d$ is invertible a linear map; and, by hypothesis, its restriction to V is well-defined (and still invertible). So to $\vec{v} \in V$ there exists $\vec{x} \in V$ with $\vec{v} = \vec{x} + T^\dagger \cdot T \cdot \vec{x}$. Let $\vec{y} := -T \cdot \vec{x} \in W$, hence $T^\dagger \vec{y} \in V$ (hypothesis): $(-T^\dagger \vec{y}, \vec{y}) \in X$; and $\in \neg B$ by Example 47a). Therefore

$$(\vec{v}, 0) = (\vec{x} + T^\dagger \cdot T \cdot \vec{x}, 0) = (\vec{x}, T \cdot \vec{x}) + (-T^\dagger \vec{y}, \vec{y}) \in (B \wedge X) \vee (\neg B \wedge X)$$

shows $V \oplus 0 \subseteq (B \wedge X) \vee (\neg B \wedge X)$. $0 \oplus W \subseteq (B \wedge X) \vee (\neg B \wedge X)$ can be seen analogously.

- b) Consider the ansatz $X = V \oplus W$ with $V, W \in \text{Gr}(\mathbb{F}^d)$, necessary for $X = (A \wedge X) \vee (\neg A \wedge X)$. Now by hypothesis and with $\neg B$ according to Example 47a), it follows

$$V \oplus 0 \subseteq X = (B \wedge X) \vee (\neg B \wedge X) = \{(\vec{x} - T^\dagger \cdot \vec{y}, T \cdot \vec{x} + \vec{y}) : \vec{x}, T^\dagger \cdot \vec{y} \in V; \vec{y}, T \cdot \vec{x} \in W\}$$

and thus $(\vec{y} = -T \cdot \vec{x}) V \subseteq \{\vec{x} + T^\dagger \cdot T \cdot \vec{x} : \vec{x} \in V, T \cdot \vec{x} \in W\}$. The condition “ $T \cdot \vec{x} \in W$ ” reveals $T \cdot V \subseteq W$. Similarly, $0 \oplus W \subseteq X$ yields $T^\dagger \cdot W \subseteq V$.

- c) Starting with the case $d = 2$ and the unitary matrix $U_2 := \begin{pmatrix} +1 & +1 \\ -1 & +1 \end{pmatrix} / \sqrt{2}$, inductively define the unitary $(d+1) \times (d+1)$ matrix $U_{d+1} := (U_d \oplus \text{id}_1) \cdot (\text{id}_{d-1} \oplus U_2)$ mapping each ‘pure’ vector $\vec{z}_i := (0^{i-1}, 1, 0^{d-i})$ to a ‘mixed’ one. Thus, the 1D subspaces $\vec{z}_i \cdot \mathbb{F}$ ($i = 1, \dots, d$) are distinct from the 1D subspaces $U \cdot \vec{z}_j \cdot \mathbb{F}$ ($j = 1, \dots, d$). More precisely, for $0 < k < d$, any subspace spanned by k among $(\vec{z}_1, \dots, \vec{z}_d)$ is distinct from any subspace spanned by k among $(U \cdot \vec{z}_1, \dots, U \cdot \vec{z}_d)$.

Now for $T := \text{diag}(1, 2, \dots, d)$, $T^\dagger T$ has as normalized eigenvectors precisely the ‘pure’ \vec{z}_i ; whereas $S := U \cdot T \cdot U^\dagger$ has only ‘mixed’ vectors $U \cdot \vec{z}_i$ as eigenvectors. Moreover every eigenspace of S is 1D and spanned by some \vec{z}_i ; whereas every eigenspace of T is spanned by some $U \cdot \vec{z}_i$. In particular, S and T have no eigenspace (even to different eigenvalues) in common. Even more, they have no non-trivial invariant subspace in common: As a special case of the uniqueness of the Jordan decomposition, any k -dimensional such subspace would be spanned by certain k eigenvectors of S as well as by certain k eigenvectors of T . But by the above considerations, two such k -dimensional spans must be distinct. \square

In the case $d = e = \text{rank}(T)$, necessarily $W = T \cdot V$; and Lemma 87a+b) establishes a one-to-one correspondence between subspaces X (necessarily of some even dimension $2k$) commuting with both $\text{graph}(0)$ and $\text{graph}(T)$ on the one hand and invariant subspaces $V \in \text{Gr}_k(\mathbb{F}^d)$ of $T^\dagger \cdot T$ on the other hand. With $X = V \oplus (T \cdot V)$, also $\neg X = \neg V \oplus \neg(T \cdot V) = \neg V \oplus ((T^{-1})^\dagger \cdot \neg V)$ is a solution to Equation (26). Indeed, since selfadjoint $H := T^\dagger \cdot T$ admits an orthogonal basis of eigenvectors, both V and $\neg V$ are invariant under H as well as under $H^{-1} = (T^{-1})^\dagger \cdot T^{-1}$.

Proof (Theorem 86).

- a) Let \vec{X} be a [weakly] satisfying assignment for f . If \vec{X} is irreducible, we are done; so suppose $Z \in \text{Gr}(\mathbb{F}^d) \setminus \{0, 1\}$ commutes with all X_i . Consider $\vec{X} \wedge Z := (X_1 \wedge Z, \dots, X_n \wedge Z)$ and $\vec{X} \wedge \neg Z$. By (the proof of) Lemma 36, it follows that $\Xi_{\mathbb{F}^d}(f; \vec{X}) = \Xi_Z(f; \vec{X} \wedge Z) \oplus \Xi_{\neg Z}(f; \vec{X} \wedge \neg Z)$. Hence if the left hand side is \mathbb{F}^d (or nonzero), then so is (at least one of the two terms on) the right hand side. Now according to Lemma 40a+b), $\vec{X} \wedge Z \in \text{Gr}(Z)$ is unitarily equivalent to some $\vec{Y} \in \text{Gr}(\mathbb{F}^e)$, $0 < e := \dim(Z) < d$; and $\vec{X} \wedge \neg Z \in \text{Gr}(\neg Z)$ to some n -tuples in $\text{Gr}(\mathbb{F}^{d-e})$ with $0 < d-e < d$. In either case, the new [weakly] satisfying assignment of f we have obtained lives in a dimension strictly less than d . So by repetition, we eventually arrive at an irreducible one.
- c) By an unitary transformation, we may w.l.o.g. presume $A = \mathbb{F}^d \times 0^d$. And by Example 47b), $B = \text{graph}(-T)$ for some $T \in \text{GL}(\mathbb{F}^d)$. Recall from Fact 4 that $C(X, A) = 1$ is equivalent to $X = c(X, A) = (A \wedge X) \vee (\neg A \wedge X)$. Recall also the aforementioned bijection due to Lemma 87a+b) between $X \in \text{Gr}_{2k}(\mathbb{F}^{2d})$ commuting with both A and B and invariant k -dimensional eigenspaces of $T^\dagger \cdot T$. Since $T^\dagger \cdot T$ is self-adjoint, it admits a spectral decomposition, i.e. a system $\vec{v}_1, \dots, \vec{v}_d \in \mathbb{F}^d$ of pairwise orthogonal eigenvectors which in particular give rise to invariant subspaces $V_i = \mathbb{F} \cdot \vec{v}_i$. If two distinct 1D eigenspaces V_i, V_j belong to the same eigenvalue $\lambda_i = \lambda_j$, then each linear combination $\mu \vec{v}_1 + \vec{v}_2$, $\mu \in \mathbb{F}$, gives rise to another 1D invariant subspace. Whereas in case that $T^\dagger \cdot T$ is *non-degenerate*, i.e. its eigenvalues $\lambda_1, \dots, \lambda_d$ (repeated according to their multiplicity) are pairwise distinct, every k -dimensional subspace is a unique sum of k among V_1, \dots, V_d ; cmp. the proof of Lemma 87c).
- b) Recall that a Boolean A commutes with everything, including $X := B$; and similarly for $B \in \{0, 1\}$. So from now on suppose $0 < A, B < 1$. If $A \vee B \neq 0$ holds, then $X := A \vee B$ has $0 < \dim(X) < d$ and $C(X, A) = 1 = C(X, B)$. Moreover, since commutativity is invariant

under complement, the same holds in cases $A \vee \neg B \neq 0$, $\neg A \vee B \neq 0$, and $\neg A \vee \neg B \neq 0$. If d is odd, exactly one of A , $\neg A$ has dimension $< d/2$; and similarly for B , $\neg B$: therefore, at least one of the above four cases must apply. By de Morgan, it remains the case that A, B are generic—which can occur only if d is even. But here, c) strikes.

- d) The first part follows from Theorem 35a); cmp. Lemma 34. For the second part, let $S, T \in \text{GL}(\mathbb{F}^d)$ according to Lemma 87c). By (the proof of) Item c), any $X \in \text{Gr}(\mathbb{F}^{2d})$ commuting with both $A := \mathbb{F}^d \times 0^d$ and $\text{graph}(-T) \in \text{Gr}_d(\mathbb{F}^{2d})$ necessarily has even $\dim(X) = 2k$ and corresponds uniquely to a k -dimensional invariant subspace of $T^\dagger T$; similarly for Y commuting with both A and $\text{graph}(-S) \in \text{Gr}_d(\mathbb{F}^{2d})$. So $X = Y$ commuting with all three A , $\text{graph}(-T)$, and $\text{graph}(-S)$ corresponds to a subspace invariant under both $T^\dagger T$ and $S^\dagger S$; which by choice of S, T means $k \in \{0, d\}$: contradiction. \square

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